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## THE GENERAL EQUATIONS OF THE ELECTRIC CIRCUIT

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*Introduction.* The following investigation of the general equations of the electric circuit was carried out during the last year, and necessarily is still very incomplete, and in some instances may require modification. As, however, our incomplete knowledge of the phenomena resulting from the stored energy of the electric field is at present the main hindrance to a practically unlimited extension of high potential transmission and distribution, the study of these phenomena represents the most important problem of electrical engineering, and the following outline is given in the hope of inducing other investigators to take up and continue the study of the subject.

As in all theoretical investigations, the most important part is the practical interpretation of the mathematical equations. Some such interpretations have been attempted throughout the following and may be read even without following the equations from which they are derived.

This paper represents an attempt to investigate mathematically the phenomena which may occur in the most general case of an electric circuit.

The general equations of the electric circuit, *i.e.*, equations which represent current and voltage in any circuit or circuit section irrespective of the character of power, voltage etc., can be derived under the condition that the constants of the electric circuit: resistance, inductance, conductance, and capacity are constant.

These equations are given by a sum of terms or groups containing trigonometrical and exponential functions of time and

distance, and the differences between all the phenomena which may occur in electric circuits merely result from different numerical values of the integration constants of this general equation.

Each group or term comprises four waves of the same frequency—two propagate in the one, the other two in the opposite direction—and can be considered as comprising two component waves, each consisting of a wave and its reflected wave, or return wave.

One of the component waves increases in the direction of propagation, the other decreases. The former, however, decreases with the time more rapidly than the latter.

A standing wave is a wave in which the component waves coincide. A standing wave has the same intensity throughout the circuit. It starts at maximum value and dies out with the time at a rate corresponding to the dissipation of energy in the circuit, as given by the time constant of the circuit.

The general wave, comprising two component waves (each consisting of main wave and return wave), gradually builds up to a maximum and then dies out again. One of the component waves dies out slower, the other faster than corresponds to the energy dissipation in the circuit, and the former decreases while the latter increases with the distance.

A free oscillation of a circuit is a wave in which no energy is supplied to or abstracted from the circuit.

The free oscillation of a uniform circuit is a standing wave, but a standing wave is not necessarily a free oscillation.

The electric wave in a uniform circuit is oscillatory in time if its frequency is high and the circuit of moderate length. With very low frequency or extreme length, as in submarine cables, or long distance telephone lines, waves may occur which are not oscillatory, but logarithmic.

In overhead transmission lines and underground cable systems all phenomena which may result in excessive voltages or currents are oscillatory in nature.

In a traveling wave each component wave traverses the circuit at constant velocity and wave shape, and an amplitude which decreases with the distance traveled, but at any point rises from zero to a maximum, and then gradually decreases again to zero. It is not a free oscillation, but energy is transferred along the circuit by it.

The phenomena of alternating currents are those of a general



wave in which the time decrement of the first component wave vanishes, that is, the energy transfer constant  $f$  equals the time constant  $w$ .

In a complex circuit, that is, circuit containing a number of sections of different constants, the same general equation can be extended throughout the entire circuit and across the transition points between different sections of the circuit of different constants, by using as distance variable the velocity of propagation of the wave in each section of the circuit.

The free oscillation of a complex circuit is not a standing wave, and the equations of the standing wave do not apply to it, but is a general wave in which the intensity of the wave increases or decreases with the distance in each section, and energy transfer occurs between the different sections of the circuit.

The free oscillation of a complex circuit dies out at a rate given by the mean time decrement, or mean time constant of the entire circuit.

Those sections having an individual time constant greater than the mean time constant of the entire circuit receive energy from the adjoining sections, while those sections having an individual time constant less than the mean time constant of the entire circuit supply energy to the rest of the circuit during a free oscillation.

During a free oscillation of a complex circuit each section of the circuit does not perform a free oscillation.

In a section of high energy dissipation, as a transmission line, the oscillation may last very much longer, and be of lower frequency, if this section is a part of a complex oscillating circuit, than if the section oscillates alone.

In the free oscillation of a complex circuit the entire length of the circuit, when expressed in velocity measure, represents a complete wave, or a half wave, or a quarter wave, or a multiple thereof.

In the free oscillation of a complex circuit the energy of the electric field of each section divides between the energy dissipated in the section and the energy transferred to other sections (or received) in a proportion depending upon the circuit constants.

At a transition point between sections of a complex circuit, the general equations remain the same when expressed with the propagation velocity as distance unit, but the numerical values of the integration constants change.

At a transition point between sections of a complex circuit transformation of current and voltage, reflection and refraction occurs.

A single wave at a transition point is partly reflected, partly transmitted, but without change of phase angle, that is, the reflected and the transmitted wave have at the transition point the same phase angle as the incident wave.

A single wave at a transition point divides between transmitted and reflected wave in a proportion depending upon the circuit constants.

The greater the change of constants at the transition point, the greater is the reflected and less the transmitted wave.

In a general wave, at a transition point, a transformation of current and voltage occurs by a ratio proportional to the circuit

constant  $\sqrt{\frac{L}{C}}$ .

When traversing a transition point a refraction occurs, so that the ratio of the tangents of the distance phase angle of the wave at the two sides of the transition point is constant and equal to the voltage transformation ratio.

### I. DERIVATION OF GENERAL EQUATIONS

1. The energy relations of an electric circuit are characterized by the four constants:

$r$  = effective resistance, representing the power consumption depending upon the current,  $i^2 r$ , or the energy voltage,  $i r$ , consumed in the circuit.

$L$  = effective inductance, representing the energy storage depending upon the current,  $\frac{i^2 L}{2}$ , as electromagnetic component,  $i L$ , of the electric field; or the voltage induced by a change of the current,  $L \frac{di}{dt}$ .

$g$  = effective shunted conductance, representing the power consumption depending upon the voltage,  $e^2 g$ , or the energy current,  $e g$ , consumed in the circuit.

$C$  = effective capacity, representing the energy storage depending upon the voltage,  $\frac{e^2 C}{2}$ , as electrostatic component,  $e C$ ,

of the electric field, or the current consumed by a change of the voltage,  $C \frac{d e}{d t}$ .

As every circuit element consumes and stores energy, and so contains all four circuit constants, the assumption usually made in the investigation of electric circuits, that these four constants  $r$ ,  $L$ ,  $g$ ,  $C$ , are localized, is permissible only in cases dealing with the usual phenomena of direct and of alternating currents, but fails when dealing with the transient phenomena resulting from the readjustment of the circuit to changes of circuit conditions, to external influences such as lightning, or when investigating high frequency phenomena, voltage and current distribution in long distance high potential circuits, cables, telephone lines, etc., and  $r$ ,  $L$ ,  $g$ ,  $C$ , must then be treated as distributed throughout the circuit.

As  $r$ ,  $L$ ,  $g$ ,  $C$ , are denoted the effective resistance, inductance, conductance, and capacity, per unit length of circuit, choosing as unit of length of the circuit whatever may be suitable, a centimeter in the high frequency oscillation over the multigap lightning-arrester circuit, or a mile in a long distance transmission circuit, or high potential cable.

In the most general case of an electric circuit or circuit section, it is possible to derive from the constants of the circuit,  $r$ ,  $L$ ,  $g$ ,  $C$ , and without any assumption whatever regarding current, voltage, etc., general equations of the electric circuit, and some results and conclusions from such equations, which are based on the single assumption, that the constants  $r$ ,  $L$ ,  $g$ ,  $C$ , remain constant with the time,  $t$ , and distance,  $u$ ; *i.e.*, are the same for every unit length of the circuit or section of the circuit to which the equations apply. Where the circuit constants change, as where another circuit joins it, the integration constants in the equations also change correspondingly.

Special cases of these general equations then are all the phenomena of direct currents, alternating currents, discharges of reactive coils, high frequency oscillations, traveling waves, etc., and the differences between these different circuits are due merely to different values of the integration constants.

2. In a circuit, or a section of a circuit, containing distributed resistance, inductance, conductance, and capacity, as a transmission line, cable, high potential coil of transformer, telephone or telegraph circuit, etc., let:

$r$  = effective resistance, per unit length of circuit.

$L$  = effective inductance, per unit length of circuit.

$g$  = effective shunted conductance, per unit length of circuit.

$C$  = effective capacity, per unit length of circuit.

Let the time be denoted by  $t$ , and the distance from some starting point by  $u$ .

Let  $e$  = voltage, and  $i$  = current, at any point  $u$  and at any time  $t$ .

$e$  and  $i$  are functions of time  $t$  and distance  $u$ .

For any element  $du$  of the circuit, the differential equations then apply:

$$\frac{de}{du} = r i + L \frac{di}{dt} \quad (1)$$

$$\frac{di}{du} = g e + C \frac{de}{dt} \quad (2)$$

Differentiating (1) after  $t$ , and (2) after  $u$ , and substituting (1) into (2), gives:

$$\frac{d^2 i}{du^2} = r g i + (r C + g L) \frac{di}{dt} + L C \frac{d^2 i}{dt^2} \quad (3)$$

and in the same manner:

$$\frac{d^2 e}{du^2} = r g e + (r C + g L) \frac{de}{dt} + L C \frac{d^2 e}{dt^2} \quad (4)$$

These differential equations, of second order, of current  $i$  and voltage  $e$ , are identical. That is:

In an electric circuit, current and electromotive force are represented by the same equations, differing by the integration constants only, which are derived from the terminal conditions of the problem.

Equation (3) is integrated by terms of the form:

$$i = A e^{-vu-st} \quad (5)$$

Substituting (5) in (3) gives the identity:

$$\begin{aligned} v^2 &= r g - (r C + g L) s + L C s^2 \\ &= (s L - r) (s C - g) \end{aligned} \quad (6)$$



In the terms of the form (5), the relation (6) must exist between the coefficients of  $u$  and  $t$ .

Substituting (6) into (1) gives:

$$\frac{d e}{d u} = (r - s L) A \epsilon^{-v u - s t} \quad (7)$$

and, integrated:

$$e = \frac{s L - r}{v} A \epsilon^{-v u - s t} \quad (8)$$

In their most general form, the *Equations of the Electric Circuit* are:

$$i = \sum_n \{A_n \epsilon^{-v_n u - s_n t}\} \quad (9)$$

$$e = \sum_n \left\{ \frac{s_n L - r}{v_n} A_n \epsilon^{-v_n u - s_n t} \right\} \quad (10)$$

$$v_n^2 = (s_n L - r) (s_n C - g) = 0 \quad (11)$$

where  $A_n$ ,  $v_n$ , and  $s_n$ , are integration constants, the latter two being related to each other by the equation (11).

3. These pairs of integration constants:  $A_n$ ,  $v_n$  and  $s_n$ , are determined by the terminal conditions of the problem, as the current and voltage at fixed points of the circuit as function of time, or the distribution of current and voltage at a fixed time as function of the distance, etc.

While  $i$  and  $e$  must always be real quantities;  $v_n$  and  $s_n$  as exponents may be complex quantities, and in this case, the integration constants,  $A_n$ , are such complex quantities, that by combining the different exponential terms of the same index  $n$ ,

the imaginary terms in  $A_n$  and  $\frac{s_n L - r}{v_n} A_n$  cancel.

Denoting, in the exponential function:

$$\epsilon^{-v u - s t}$$

$$v = h + j k \quad (12)$$

$$s = p + j q$$

it is:

$$\epsilon^{-v u - s t} = \epsilon^{-h u - p t} \epsilon^{-j(k u + q t)}$$

and the latter term resolves into trigonometric functions of the angle:

$$k u + q t = \text{constant} \quad (13)$$

so gives the relation between  $u$  and  $t$  for constant phase of the oscillation or alternation of the current or voltage.

With change of time  $t$ , the phase changes in position  $u$ , in the circuit, *i.e.*, moves along the circuit, and the velocity of this motion is given by differentiating (13) after  $t$  as

$$V = \frac{d u}{d t} = - \frac{q}{k} \quad (14)$$

that is:

$$V = - \frac{q}{k} \quad (15)$$

is the velocity of *propagation of the electric phenomena in the circuit*.

(If no energy losses occur,  $r = 0$ ,  $g = 0$ , in a straight conductor in a medium of unit magnetic and dielectric constant, that is, unit permeability and unit inductive capacity;  $V$  is the velocity of light.)

4. Since (11) is a quadratic equation, several pairs or corresponding values of  $v$  and  $s$  exist, which correspond to the same equation (11), and can be combined with each other into a group.

Such a group of terms, of the same index  $n$  is defined by the equation (11):

$$v^2 = (s L - r) (s C - g) \quad (16)$$

and substituting:

$$v = v_1 \sqrt{L C} \quad (17)$$

and also:

$$\begin{aligned} v &= h + j k \\ v_1 &= h_1 + j k_1 \\ s &= p + j q \end{aligned} \quad (18)$$

Hence:

$$\begin{aligned} h &= h_1 \sqrt{L C} \\ k &= k_1 \sqrt{L C} \end{aligned} \quad (19)$$

it is, substituting (18) in (16):

$$(h_1 + j k_1)^2 = \left[ (p + j q) - \frac{r}{L} \right] \left[ (p + j q) - \frac{g}{C} \right] \quad (20)$$

resolved, and substituting:

$$w = \frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right) \quad (21)$$

$$m = \frac{1}{2} \left( \frac{r}{L} - \frac{g}{C} \right) \quad (22)$$

$$p = f + w \quad (23)$$

gives:

$$\left. \begin{aligned} h_1^2 - k_1^2 &= f^2 - q^2 - m^2 \\ h_1 k_1 &= f q \end{aligned} \right\} \quad (24)$$

and:

$$\left. \begin{aligned} f^2 - q^2 &= h_1^2 - k_1^2 + m^2 \\ f q &= h_1 k_1 \end{aligned} \right\} \quad (25)$$

hence:

$$\left. \begin{aligned} h &= \sqrt{L C} \sqrt{\frac{1}{2} \{R_1^2 + f^2 - q^2 - m^2\}} \\ k &= \sqrt{L C} \sqrt{\frac{1}{2} \{R_1^2 - f^2 + q^2 + m^2\}} \\ R_1^2 &= \sqrt{(f^2 + q^2 - m^2)^2 + 4 q^2 m^2} = \frac{h^2 + k^2}{L C} \end{aligned} \right\} \quad (26)$$

or:

$$\left. \begin{aligned} f &= \frac{1}{\sqrt{L C}} \sqrt{\frac{1}{2} \{R_2^2 + h^2 - k^2 + L C m^2\}} \\ q &= \frac{1}{\sqrt{L C}} \sqrt{\frac{1}{2} \{R_2^2 - h^2 + k^2 - L C m^2\}} \end{aligned} \right\} \quad (27)$$

$$R_2^2 = \sqrt{(h^2 + k^2 + L C m^2)^2 - 4 L C k^2 m^2} = L C (f^2 + q^2)$$

This gives eight pairs of corresponding values of  $v$  and  $s$ :

$$\left. \begin{array}{ll} 1. \ v = +h + jk & s = w - f - j q \\ & +h - jk & w - f + j q \\ 2. \ v = -h - jk & s = w - f - j q \\ & -h + jk & w - f + j q \\ 3. \ v = -h + jk & s = w + f - j q \\ & -h - jk & w + f + j q \\ 4. \ v = +h - jk & s = w + f - j q \\ & +h + jk & w + f + j q \end{array} \right\} \quad (28)$$

Substituting these values in the general equations (9) and (10), and in the usual manner transforming the exponential functions of imaginary exponents, into trigonometric functions, and substituting for the integration constants,  $A$ , the new constants:

$$C = A + A'$$

$$C' = j (A - A')$$

then gives as the general expression of the

*Equations of the Electric Circuit:*

$$\begin{aligned} i &= \Sigma [\epsilon^{-hu-(w-f)t} \{C_1 \cos (qt - ku) + C_1' \sin (qt - ku)\}] \quad (i_1) \\ &\quad - \epsilon^{+hu-(w-f)t} \{C_2 \cos (qt + ku) + C_2' \sin (qt + ku)\} \quad (i_2) \\ &\quad + \epsilon^{+hu-(w+f)t} \{C_3 \cos (qt - ku) + C_3' \sin (qt - ku)\} \quad (i_3) \\ &\quad - \epsilon^{-hu-(w+f)t} \{C_4 \cos (qt + ku) + C_4' \sin (qt + ku)\} \quad (i_4) \end{aligned} \quad (29)$$

$$\begin{aligned} e &= \Sigma [\epsilon^{-hu-(w-f)t} \{(c_1' C_1' - c_1 C_1) \cos (qt - ku) - (c_1' C_1 + c_1 C_1') \sin (qt - ku)\}] \quad (e_1) \\ &\quad + \epsilon^{+hu-(w-f)t} \{(c_1' C_2' - c_1 C_2) \cos (qt + ku) - (c_1' C_2 + c_1 C_2') \sin (qt + ku)\} \quad (e_2) \\ &\quad + \epsilon^{+hu-(w+f)t} \{(c_2' C_3' - c_2 C_3) \cos (qt - ku) - (c_2' C_3 + c_2 C_3') \sin (qt - ku)\} \quad (e_3) \end{aligned} \quad (30)$$

$$+ \varepsilon^{-h u - (w+f)t} \{ (c_2' C_4' - c_2 C_4) \cos (qt + ku) - (c_2' C_4 + c_2 C_4') \sin (qt + ku) \} \quad (e_4)$$

where  $C_1, C_1', C_2, C_2', C_3, C_3', C_4, C_4'$ , and two of the four values,  $f, q, h, k$ , are integration constants, depending on the terminal conditions, and:

$$\left. \begin{aligned} c_1 &= \frac{qk + h(m+f)}{h^2 + k^2} L \\ c_1' &= \frac{k(m+f) - qh}{h^2 + k^2} L \\ c_2 &= \frac{qk - h(m-f)}{h^2 + k^2} L \\ c_2' &= \frac{k(m-f) + qh}{h^2 + k^2} L \end{aligned} \right\} \quad (31)$$

and:

$$\left. \begin{aligned} w &= \frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right) \\ m &= \frac{1}{2} \left( \frac{r}{L} - \frac{g}{C} \right) \end{aligned} \right\} \quad (32)$$

and  $h, k$ , and  $f, q$ , are related by the equations:

$$\left. \begin{aligned} h &= \sqrt{LC} \sqrt{\frac{1}{2} \{ R_1^2 + f^2 - q^2 - m^2 \}} \\ k &= \sqrt{LC} \sqrt{\frac{1}{2} \{ R_1^2 - f^2 + q^2 + m^2 \}} \\ R_1^2 &= \sqrt{(f^2 + q^2 - m^2)^2 + 4q^2 m^2} \end{aligned} \right\} \quad (33)$$

hence:

$$h^2 + k^2 = LC R_1^2 \quad (34)$$

or:

$$\left. \begin{aligned} f &= \frac{1}{\sqrt{LC}} \sqrt{\frac{1}{2} \{ R_2^2 + h^2 - k^2 + LC m^2 \}} \\ q &= \frac{1}{\sqrt{LC}} \sqrt{\frac{1}{2} \{ R_2^2 - h^2 + k^2 - LC m^2 \}} \\ R_2^2 &= \sqrt{(h^2 + k^2 + LC m^2)^2 - 4LC k^2 m^2} \end{aligned} \right\} \quad (35)$$



$$f^2 + q^2 = \frac{R_1^2}{L C} \quad (36)$$

## II. DISCUSSION OF GENERAL EQUATIONS.

5. The general equations of current and voltage of a circuit or section of circuit having uniformly distributed and constant values of  $r$ ,  $L$ ,  $g$ ,  $C$ , appear as a sum of groups of four terms each, characterized by the feature, that the four terms of each group have the same values of  $f$ ,  $q$ ,  $h$ ,  $k$ .

Of the four terms of each group,  $i_1$ ,  $i_2$ ,  $i_3$ ,  $i_4$ , or  $e_1$ ,  $e_2$ ,  $e_3$ ,  $e_4$ , respectively (equations (29) and (30)), two contain the angles  $(qt - ku)$ ;  $i_1$ ,  $e_1$ , and  $i_3$ ,  $e_3$ ; and two contain the angles  $(qt + ku)$ ;  $i_2$ ,  $e_2$ , and  $i_4$ ,  $e_4$ .

In the former, the velocity of propagation is:

$$\frac{du}{dt} = + \frac{q}{k}$$

hence positive, that is, towards raising  $u$ .

In the latter, the velocity of propagation is:

$$\frac{du}{dt} = - \frac{q}{k}$$

or negative, that is, towards decreasing  $u$ .

Each group consists of two waves and their reflected waves.

$i' = i_1 - i_2$  and  $e' = e_1 + e_2$  is the first wave and its reflected wave, and

$i'' = i_3 - i_4$  and  $e'' = e_3 + e_4$  is the second wave and its reflected wave.

In the first wave,  $i'$ ,  $e'$ , the amplitude decreases in the direction of propagation,  $\epsilon^{-ku}$  for rising,  $\epsilon^{+ku}$  for decreasing  $u$ , and the wave dies out with increasing time  $t$ , by  $\epsilon^{-(wt-t)} = \epsilon^{-wt} \epsilon^{+ft}$ .

In the second wave,  $i''$ ,  $e''$ , the amplitude increases in the direction of propagation,  $\epsilon^{+ku}$  for rising,  $\epsilon^{-ku}$  for decreasing  $u$ , but the wave dies out with the increasing time  $t$ , by  $\epsilon^{-(wt+f)t} = \epsilon^{-wt} \epsilon^{-ft}$ , that is, faster than the first wave.

If the amplitude of the wave remained constant throughout the circuit—as would be the case in a free oscillation of the circuit, in which the stored energy of the circuit is dissipated,

but no power transferred one way or the other—that is, if  $h=0$ , it also is, by (56),  $f=0$ , that is, both waves coincide into one, which dies out with the time by the decrement  $\epsilon^{-wt}$ .

It thus follows:

In general, two waves, with their reflected waves, traverse the circuit, of which the one  $i''$ ,  $e''$ , increases in amplitude in the direction of propagation, but dies out correspondingly more rapidly in time, that is, faster than a wave of constant amplitude; while the other  $i'$ ,  $e'$ , decreases in amplitude, but lasts a longer time, that is, dies out slower than a wave of constant amplitude. That is, in the one wave,  $i''$ ,  $e''$ , a decrease of amplitude takes place at a sacrifice of duration in time, while in the other wave,  $i'$ ,  $e'$ , a slower dying out of the wave with the time is produced at the expense of a decrease of amplitude during its propagation.

It is seen herefrom, that the equations usually given for an oscillating circuit, which do not contain an exponential function of the distance, are not the most general equations, but correspond only to the special case of a free oscillation in which no energy transfer takes place, and so do not apply to the traveling wave or impulse, nor to the oscillation of a complex circuit.

6. In the equations (29) and (30):

$$q t = 2 \pi$$

gives the time of a complete cycle, that is, the *period of the wave*:

$$T = \frac{2\pi}{q}$$

and:

$$N = \frac{q}{2\pi}$$

the *frequency of the wave*.

$$k u = 2 \pi$$

gives the distance of a complete cycle, that is, the *wave length*:

$$U = \frac{2\pi}{k}$$

$$(w - f) t = 1 \text{ and}$$

$$(w + f) t = 1$$

gives the time:

$$T_1' = \frac{1}{w-f} \text{ and}$$

$$T_1'' = \frac{1}{w+f}$$

during which the wave decreases to  $1/\epsilon = .3679$  of its value, and

$$h u = 1$$

gives the distance:

$$U_1 = 1/h$$

over which the wave decreases to  $1/\epsilon = .3679$  of its value.

That is:

$q$  is the *frequency constant* of the wave:

$k$  is the *wave length constant*:

$(w-f)$  and  $(w+f)$  are the *time attenuation constants* of the wave:

$h$  is the *distance attenuation constant* of the wave:

7. If the frequency of the current and voltage is very high, thousands of cycles and more, as with traveling waves, lightning disturbances, high frequency oscillations, etc.,  $q$  is a very large quantity compared with  $f$ ,  $w$ ,  $m$ ,  $h$ ,  $k$ , and  $k$  is a large quantity compared with  $h$ , and, by dropping terms of secondary order, the preceding equations can be simplified, and become:

$$\left. \begin{aligned} h &= f \sqrt{L C} \\ k &= q \sqrt{L C} \end{aligned} \right\} \quad (37)$$

and:

$$\left. \begin{aligned} c_1 &= c_2 = \sqrt{\frac{L}{C}} \\ c_1' &= c_2' = \frac{m}{q} \sqrt{\frac{L}{C}} = 0 \end{aligned} \right\} \quad (38)$$

Denoting:

$$\left. \begin{aligned} \sigma &= \sqrt{L C} \\ c &= \sqrt{\frac{L}{C}} \end{aligned} \right\} \quad (39)$$

where  $\sigma$  is the reciprocal of the frequency of propagation (velocity of light), it is:

$$\left. \begin{aligned} h &= \sigma f \\ k &= \sigma q \end{aligned} \right\} \quad (40)$$

$$\left. \begin{aligned} c_1 &= c_2 = c \\ c_1' &= c_2' = \frac{m}{q} c = 0 \end{aligned} \right\} \quad (41)$$

and, introducing the new independent variable, as distance:

$$\lambda = \sigma u \quad (42)$$

it is:

$$\left. \begin{aligned} k u &= q \lambda \\ h u &= f \lambda \end{aligned} \right\} \quad (43)$$

hence, the wave length is given by:

$$q \lambda = 2 \pi$$

as:

$$\lambda_0 = \frac{2\pi}{q} \quad (44)$$

and since the period is:

$$T_0 = \frac{2\pi}{q}$$

it follows, that by the introduction of the denotation (42), distances are measured with the velocity of propagation as unit length, and wave length  $U$  and period  $T$  have the same numerical values.

Substituting now in equations (29) and (30), gives:

$$\begin{aligned} i &= \epsilon^{-wt} \sum \{ \epsilon^{+f(t-\lambda)} D_1 [q(t-\lambda)] - \epsilon^{+f(t+\lambda)} D_2 [q(t+\lambda)] \\ &\quad + \epsilon^{-f(t-\lambda)} D_3 [q(t-\lambda)] - \epsilon^{-f(t+\lambda)} D_4 [q(t+\lambda)] \} \end{aligned} \quad \begin{matrix} (i') \\ (i'') \end{matrix} \quad (45)$$

$$\begin{aligned} e &= \sqrt{\frac{L}{C}} \epsilon^{-wt} \sum \{ \epsilon^{+f(t-\lambda)} D_1 [q(t-\lambda)] + \epsilon^{+f(t+\lambda)} D_2 [q(t+\lambda)] \\ &\quad + \epsilon^{-f(t-\lambda)} D_3 [q(t-\lambda)] + \epsilon^{-f(t+\lambda)} D_4 [q(t+\lambda)] \} \end{aligned} \quad \begin{matrix} (e') \\ (e'') \end{matrix} \quad (46)$$

where:

$$D [q (t \pm \lambda)] = C \cos q (t \pm \lambda) + C' \sin q (t \pm \lambda) \quad (47)$$

8. As seen from equations (45) and (46), the waves are products of  $\epsilon^{-wt}$  and a function of  $(t - \lambda)$  for the main wave,  $(t + \lambda)$  for the reflected wave:

$$\left. \begin{aligned} i_1 + i_3 &= \epsilon^{-wt} f_1 (t - \lambda) \\ i_2 + i_4 &= \epsilon^{-wt} f_2 (t + \lambda) \end{aligned} \right\} \quad (48)$$

Hence, for constant  $(t - \lambda)$  on the main waves, and for constant  $(t + \lambda)$  on the reflected waves, it is:

$$\left. \begin{aligned} i_1 + i_3 &= B \epsilon^{-wt} \\ i_2 + i_4 &= B' \epsilon^{-wt} \end{aligned} \right\} \quad (49)$$

That is, during its passage along the circuit, the wave decreases by the decrement  $\epsilon^{-wt}$ , or at a constant rate, independent of frequency, wave length, etc., and depending merely on the circuit constants  $r, L, g, C$ .

That is, the decrement of the traveling wave in the direction of its motion is independent of the character of the wave, as its frequency, etc.

In other words, an impulse traveling along the circuit retains its wave shape, provided that  $r, L, g, C$ , are constant.

If  $r$  and  $g$  increase with the frequency, as is often the case, the higher frequency waves die out faster, and the wave shape gradually smooths out in the direction of travel.

The physical meaning of the two waves  $i'$  and  $e''$  can best be appreciated by observing the effect of the wave when traversing a fixed point  $\lambda$  of the circuit.

Consider as instance the main wave only;  $i' = i_1 + i_3$ ; the same applies for the reflected wave.

Assuming then, that at time  $t=0$ , the current  $I'=0$ , it is, for constant  $\lambda$ :

$$I = D (\epsilon^{-(w-\lambda)t} - \epsilon^{-(w+\lambda)t})$$

the amplitude of current  $I$  at point  $\lambda$ .

As this is the difference of two exponential functions of different decrement, it follows, that as function of time  $t$ ,  $I'$  rises from 0 to a maximum and then decreases again to zero, as

shown in Fig. 1, and the actual current  $i$  is the oscillatory wave with  $I$  as envelope.

The combination of the two waves thus represents the passage

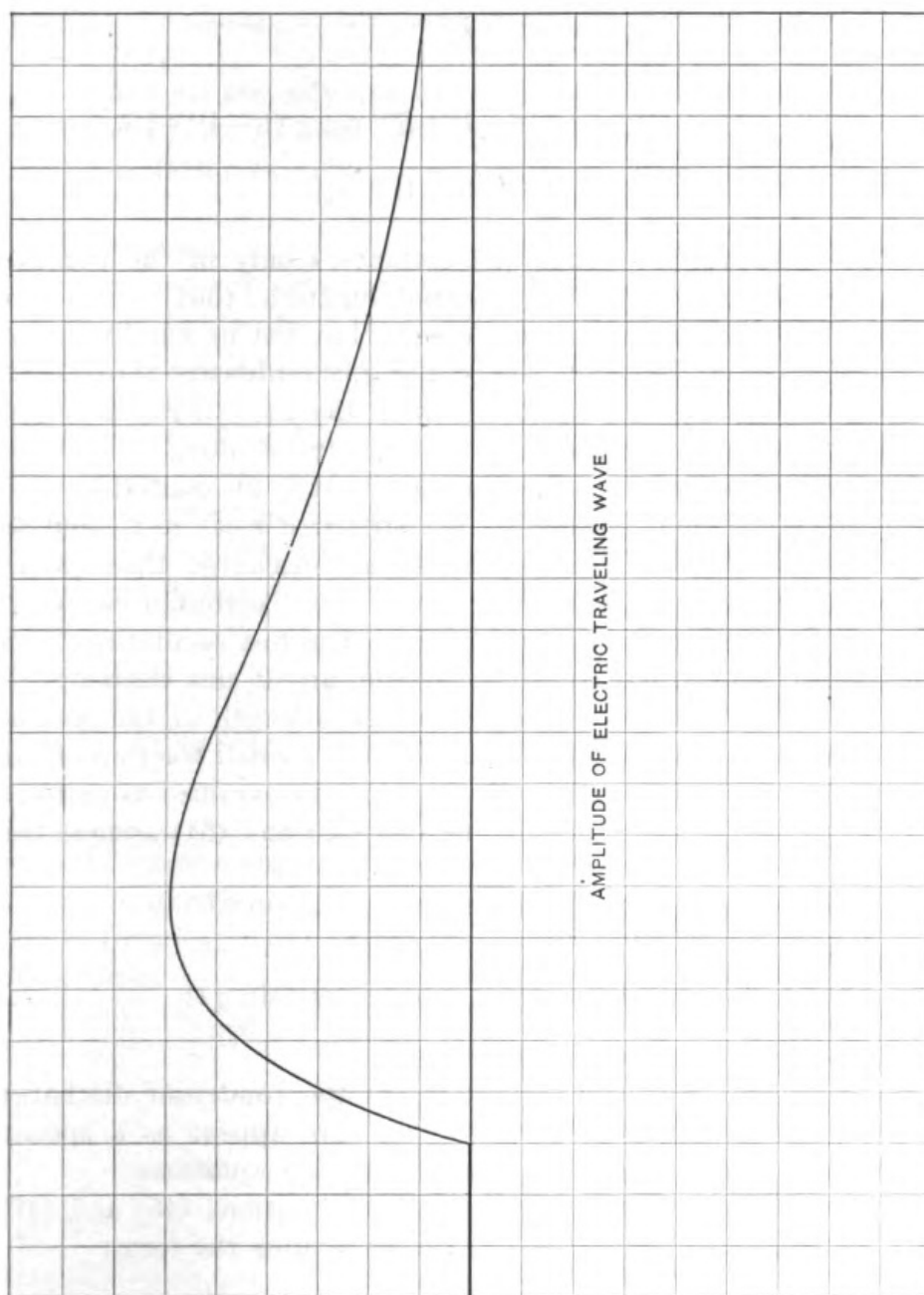


FIG. 1

of a wave across a given point, the amplitude rising during the arrival, and decreasing again after the passage of the wave.

9. If  $h$  and also  $f$  equal zero,  $i'$ ,  $e'$ , and  $i''$ ,  $e''$ , coincide in



equations (45) and (46), and the equations (45) (46) assume the form:

$$i = \epsilon^{-wt} \Sigma \{ [B_1 \cos q(t - \lambda) + B_1' \sin q(t - \lambda)] - [B_2 \cos q(t + \lambda) + B_2' \sin q(t + \lambda)] \} \quad (50)$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \Sigma \{ [B_1 \cos q(t - \lambda) + B_1' \sin q(t - \lambda)] + [B_2 \cos q(t + \lambda) + B_2' \sin q(t + \lambda)] \} \quad (51)$$

These equations contain the distance  $\lambda$  only in the trigonometric, but not in the exponential function, that is,  $i$  and  $e$  vary in phase throughout the circuit, but not in amplitude, or in other words, the oscillation is of uniform intensity throughout the circuit, dying out uniformly with the time, from an initial maximum value, but the wave does not travel along the circuit, but is a stationary or standing wave. It is an oscillatory discharge of a circuit containing a distributed  $r$ ,  $L$ ,  $g$ ,  $C$ , and so analogous to the oscillating condenser discharge through an inductive circuit, except that, due to the distributed capacity, the phase changes along the circuit. The free oscillations of a uniform circuit, as a transmission line, are of this character.

For  $\lambda = 0$ , that is, assuming the wave length of the oscillation is so great, hence the circuit as such a small fraction of the wave length, that the phase of  $i$  and  $e$  can be assumed as uniform throughout the circuit, the equations (50) and (51) assume the form:

$$i = \epsilon^{-wt} \{ B_0 \cos q t - B_0' \sin q t \}$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ B_0 \cos q t + B_0' \sin q t \}$$

and these are the usual equations of the condenser discharge through an inductive circuit, which here appear as a special case of a special case of the general circuit equations.

If  $q$  equals zero, the functions  $D$  in equations (45) and (46) become constant, and these equations assume the form:

$$i = \epsilon^{-wt} \Sigma \{ [C_1 \epsilon^{+f(t-\lambda)} + C_3 \epsilon^{-f(t-\lambda)}] - [C_2 \epsilon^{+f(t+\lambda)} + C_4 \epsilon^{-f(t+\lambda)}] \}$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \Sigma \{ [C_1 \epsilon^{+f(t-\lambda)} + C_3 \epsilon^{-f(t-\lambda)}] + [C_2 \epsilon^{+f(t+\lambda)} + C_4 \epsilon^{-f(t+\lambda)}] \}$$

This gives expressions of current and voltage, which are not oscillatory any more, but exponential, representing a gradual change of  $i$  and  $e$  as functions of time and distance, corresponding to the gradual or logarithmic condenser discharge. For  $\lambda = 0$ , these equations change to the equations of the logarithmic condenser discharge.

These equations are only approximate however, since in them the quantities  $f$ ,  $w$ ,  $h$ , have been neglected compared with  $q$ , assuming the latter as very large, while now it is assumed as zero.

It thus follows:

If the constant  $h$  in equations (29) and (30) differs from zero, the oscillation (using the term *oscillation* here in the most general sense, that is, including also *alternation*, as an oscillation of zero attenuation) travels along the circuit; but it becomes stationary, as standing wave, for  $h = 0$ . That is:

The distance attenuation constant  $h$  may also be called the *propagation* constant of the wave.

$h = 0$  thus represents a wave which does not propagate or move along the circuit, but stands still, that is, a stationary or standing wave.

### III. STANDING WAVES.

10. If the propagation constant of the wave vanishes:

$$h = 0$$

the wave becomes a stationary or standing wave, and the equations of the standing wave so are derived from the general equations, by substituting thereon:  $h = 0$ .

This gives:

$$R_1^2 = \sqrt{(k^2 - L C m^2)^2}$$

hence, if:

$$k^2 > L C m^2: R_1^2 = k^2 - L C m^2,$$

and, if:

$$k^2 < L C m^2: R_1^2 = L C m^2 - k^2$$

Two different cases exist, depending upon the relative values of  $k^2$  and  $L C m^2$ , and in addition thereto the intermediary or critical case, in which  $k^2 = L C m^2$ .

These three cases require separate consideration.

$$L C m^2 = \left\{ L C \frac{1}{2} \left( \frac{r}{L} - \frac{g}{C} \right) \right\}^2$$

is a circuit constant, while  $k$  is the wave length constant, that is, the higher  $k$ , the shorter is the wave length.

A. *Short Waves*:

$$k^2 > L C m^2$$

hence:

$$R_2^2 = k^2 - L C m^2$$

and,

$$f = 0$$

and approximately,

$$q = \frac{k}{\sqrt{L C}}$$

Substituting in equations (29), (30), the two waves  $i'$ ,  $e'$ , and  $i''$ ,  $e''$ , coincide, and all the exponential terms reduce to  $\epsilon^{-wt}$ , and rearranged, give,

$$i = \epsilon^{-wt} \{ [B_1 \cos (qt - ku) + B_1' \sin (qt - ku)] - [B_2 \cos (qt + ku) + B_2' \sin (qt + ku)] \} \quad (52)$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ [B_1 \cos (qt - ku) + B_1' \sin (qt - ku)] + [B_2 \cos (qt + ku) + B_2' \sin (qt + ku)] \} \quad (53)$$

These equations represent a *stationary electrical oscillation* or standing wave in the circuit.

B. *Long Waves*:

$$k^2 < L C m^2$$

hence:

$$R_2^2 = L C m^2 - k^2$$

and:

$$f = \sqrt{m^2 - \frac{k^2}{LC}}$$

$$q = 0$$

or approximately, for very small values of  $k$ :

$$f = m = \frac{1}{2} \left( \frac{r}{L} - \frac{g}{C} \right)$$

Substituting now into (29) and (30), the two waves  $i'$ ,  $e'$ , and  $i''$ ,  $e''$ , remain separate, having different exponential terms,  $\epsilon^{-(m-f)t}$  and  $\epsilon^{-(m+f)t}$ , but in each of the two waves, the main wave and the reflected wave coincide, due to the vanishing of  $q$ .

$$i = \epsilon^{-wt} \{ (B_1 \epsilon^{+ft} + B_2 \epsilon^{-ft}) \cos ku - (B_1' \epsilon^{+ft} + B_2' \epsilon^{-ft}) \sin ku \} \quad (54)$$

$$\begin{aligned} e &= \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ [(m+f) B_1' \epsilon^{+ft} + (m-f) B_2' \epsilon^{-ft}] \cos ku + [(m+f) B_1 \epsilon^{+ft} + (m-f) B_2 \epsilon^{-ft}] \sin ku \} \\ &= \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ m [(B_1' \epsilon^{+ft} + B_2' \epsilon^{-ft}) \cos ku + (B_1 \epsilon^{+ft} + B_2 \epsilon^{-ft}) \sin ku] \\ &\quad + f [(B_1' \epsilon^{+ft} - B_2' \epsilon^{-ft}) \cos ku + (B_1 \epsilon^{+ft} - B_2 \epsilon^{-ft}) \sin ku] \} \end{aligned}$$

These equations represent a *gradual or exponential circuit discharge*. The distribution is a trigonometric function of the distance, that is, a wave distribution, but dies out gradually in time, or without oscillation.

C. *Critical Case*:

$$k^2 = LC m^2$$

This gives:

$$i = \epsilon^{-wt} \{ B \cos ku - B' \sin ku \}$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ B' \cos ku + B \sin ku \}$$

In the critical case, the wave is distributed as trigono-

metric function of the distance, but dies out as simple exponential function of the time.

11. An electrical standing wave can have two different forms: it can be either oscillatory in time, or exponential in time, that is, gradually changing. It is interesting to investigate the conditions under which these two different cases occur.

The transition from gradual to oscillatory, takes place at:

$$k_0^2 = m^2 L C$$

for larger values of  $k$ , the phenomenon is oscillatory; for smaller, exponential or gradual:

As  $k$  is the wave length constant, the wave length, at which the phenomenon ceases to be oscillatory in time, and becomes a gradual dying out, is given by:

$$U_0 = \frac{2\pi}{k_0} \\ = \frac{2\pi}{m\sqrt{LC}}$$

In an undamped wave, that is, in a circuit of zero  $r$  and zero  $g$ , in which no energy losses occur, the velocity of propagation is:

$$v = \frac{q}{k} = \frac{1}{\sqrt{LC}}$$

and is the velocity of light:

$$v_0 = 3 \times 10^{10}$$

if the medium has unit permeability and unit inductivity.

In an undamped circuit, this wave length  $U_0$  would correspond to the frequency:

$$q_0 = \frac{k_0}{\sqrt{LC}} = m$$

hence:

$$N_0 = \frac{q_0}{2\pi} = \frac{m}{2\pi}$$

The frequency, at the wave length  $U_0$ , is however, zero, as at this wave length the phenomenon ceases to be oscillatory. That is:

Due to the energy losses in the circuit, by the effective resistance  $r$  and effective conductance  $g$ , the frequency  $N$  of the wave is reduced below the value corresponding to the wave length  $U$ , the more, the greater the wave length, until at the wave length  $U_0$  the frequency becomes zero, and the phenomenon thereby non-oscillatory. This means, with increasing wave length, the velocity of propagation of the phenomenon decreases, and becomes zero at wave length  $U_0$ .

$$\text{If } m^2 L C = 0, k_0 = 0, \text{ and } U_0 = \infty,$$

that is, the standing wave is always oscillatory.

$$\text{If } m^2 L C = \infty, k_0 = \infty, \text{ and } U_0 = 0,$$

that is, the standing wave is always non-oscillatory, or gradually dying out.

In the former case:

$$m^2 L C = 0, \text{ or oscillatory phenomenon,}$$

it is, substituting for  $m^2$  and expanding:

$$\frac{r}{g} = \frac{L}{C}$$

or:

$$r C - g L = 0$$

In the latter case:

$m^2 L C = \infty$ , or non-oscillatory or exponential standing wave, it is either:

$$L = 0, \text{ or } C = 0.$$

That is:

The standing wave in a circuit is always oscillatory, regardless of its wave length, if

$$\frac{r}{g} = \frac{L}{C},$$

that is, the ratio of the energy coefficients equals the ratio of the reactive coefficients of the circuit.

The standing wave can never be oscillatory, but is always exponential, or gradually dying out, if either the inductance  $L$ , or the capacity  $C$ , vanish, that is, the circuit contains no capacity or contains no inductance. In all other cases, the standing wave is oscillatory for waves longer than the critical value  $U_0 = \frac{2\pi}{k_0}$ , where

$$k_0^2 = m^2 L C = \frac{1}{4} \left\{ r \sqrt{\frac{C}{L}} - g \sqrt{\frac{L}{C}} \right\}^2 \quad (56)$$

and is exponential or gradual, for standing waves shorter than the critical wave length  $U_0$ . Or, for  $k > k_0$ , the standing wave is exponential; for  $k < k_0$ , it is oscillatory.

The term  $k_0 = m \sqrt{L C}$  takes a similar part in the theory of standing waves as the term:  $r_0^2 - 4 L_0 C_0$  in the condenser discharge through an inductive circuit, that is, it separates the exponential or gradual, from trigonometric or oscillatory conditions.

The difference is, that the condenser discharge through an inductive circuit is gradual, or oscillatory, depending on the circuit constants, while in a general circuit, with the same circuit constants, in general gradual as well as oscillatory standing waves exist, the former at greater wave length, or

$$m \sqrt{L C} > k,$$

the latter for shorter wave length, or

$$m \sqrt{L C} < k,$$

An idea of the quantity  $k_0$ , and so the wave length  $U_0$ , at which the frequency of the standing wave becomes zero, or the wave non-oscillatory, and of the frequency  $N_0$ , which in an undamped circuit will correspond to this critical wave length  $N_0$ , can best be derived by considering some representative numerical instance.

1. *High Power High Potential Overhead Transmission Line.*

12. Assuming power to be transmitted 120 miles, at 40,000 volts between line and ground, by a three-phase system with grounded neutral, over a line consisting of copper conductors, wire No. 00 B. & S. gauge with 5 feet between conductors.



Choosing as unit length the mile, it is, approximately,

$$r = .41 \text{ ohms}$$

$$L = 1.95 \times 10^{-3} h$$

$$g = .25 \times 10^{-6} \text{ mho}$$

$$C = .0162 \times 10^{-6} f.$$

Herefrom then follows:

$$w = 113$$

$$m = 97$$

$$\sigma = \sqrt{L C} = 5.62 \times 10^{-6}$$

$$k_0 = m L C = 545 \times 10^{-6}$$

hence, the critical wave length:

$$U_0 = \frac{2\pi}{k_0} = 11,500 \text{ miles}$$

and, in an undamped circuit, this wave length would correspond to the frequency of oscillation:

$$N_0 = \frac{m}{2\pi} = 15.7 \text{ cycles per second.}$$

Since the shortest wave at which the phenomenon ceases to be oscillatory, is 11,500 miles in length, and the longest wave, which can originate in the circuit, is 4 times the length of the circuit, or 480 miles, it follows that whatever waves may originate in this circuit, are by necessity oscillatory, and non-oscillatory currents or voltages can exist in this circuit only when impressed upon it by some outside source, and then are of such great wave length, that the circuit is only an insignificant fraction of the wave, and great differences of voltage and current of non-oscillatory nature cannot exist.

Since the difference in length, between the shortest non-oscillatory wave, and the longest wave which can originate in the circuit, is so very great, it follows, that in high potential long distance transmission circuits, all phenomena which may result in considerable potential differences and differences of current throughout the circuit, are oscillatory in nature.

With a length of circuit of 120 miles, the longest standing wave, which can originate in the circuit, has the wave length:

$$U = 480 \text{ miles.}$$

and herefrom follows:

$$k = \frac{2\pi}{U} = .0134$$

$$q = \frac{k}{\sqrt{LC}} = 2380$$

$$N = \frac{q}{2\pi} = 380 \text{ cycles.}$$

Hence, even for the longest standing wave, which may originate in this transmission line, and still more so for shorter waves, the overtones of the fundamental wave, the approximations in paragraph 7 are justified, and it thus is:

*General Equations of Standing Waves in Long Distance Power Transmission Lines:*

$$i = \Sigma \epsilon^{-wt} \{ [B_1 \cos (qt - ku) + B_1' \sin (qt - ku)] - [B_2 \cos (qt + ku) + B_2' \sin (qt + ku)] \} \quad (57)$$

$$e = + \sqrt{\frac{L}{C}} \Sigma \epsilon^{-wt} \{ [B_1 \cos (qt - ku) + B_1' \sin (qt - ku)] + [B_2 \cos (qt + ku) + B_2' \sin (qt + ku)] \} \quad (58)$$

or:

$$e = \Sigma \epsilon^{-wt} \{ [A_1 \cos (qt + ku) + A_1' \sin (qt + ku)] + [A_2 \cos (qt - ku) + A_2' \sin (qt - ku)] \} \quad (59)$$

$$i = \sqrt{\frac{C}{L}} \Sigma \epsilon^{-wt} \{ [A_1 \cos (qt + ku) + A_1' \sin (qt + ku)] - [A_2 \cos (qt - ku) + A_2' \sin (qt - ku)] \} \quad (60)$$

where:

$$A_1 = -\sqrt{\frac{L}{C}} B_2$$

$$A_1' = -\sqrt{\frac{L}{C}} B_2'$$

etc.

2. *High Potential Underground Power Cable.*

13. Assuming, per mile of single conductor:

$$r = .41 \text{ ohms.}$$

$$L = .4 \times 10^{-3} h.$$

$g = 1 \times 10^{-6}$  mho, corresponding to a power factor of the cable charging current, at 25 cycles, of 1 per cent.

$$C = .6 \times 10^{-6} f.$$

Herefrom then follows:

$$w = 513$$

$$m = 512$$

$$\sigma = \sqrt{L C} = 15.5 \times 10^{-9}$$

$$k_0 = m \sqrt{L C} = 7.95 \times 10^{-3}$$

and, critical wave length:

$$U_0 = 790 \text{ miles,}$$

and, frequency of an undamped oscillation, corresponding to  $U_0$ :

$$N_0 = 81.5 \text{ cycles.}$$

As seen, in an underground high potential cable, the critical wave length is very much shorter than in the overhead long distance transmission line. At the same time, however, the length of an underground cable circuit is very much shorter than that of a long distance transmission line, so that the critical wave length still is very large compared with the greatest wave length of an oscillation originating in the cable; about 10 times as great.

That is, the discussion of the possible phenomena in any overhead line applies also to the underground high potential cable circuit.

### 3. *Submarine Telegraph Cable.*

Choosing the values:

Length of cable, 4000 miles.

Constants per mile of conductor:

$$r = 3 \text{ ohms}$$

$$L = 1 \times 10^{-3} h$$

$$g = 1 \times 10^{-6} \text{ mho}$$

$$C = .1 \times 10^{-6} f$$

hence:

$$w = 1500$$

$$m = 1500$$

$$\sigma = LC = 10 \times 10^{-6}$$

$$k_0 = m \sqrt{LC} = 15 \times 10^{-3}$$

and:

$$U_0 = 418 \text{ miles}$$

thus:

$$N_0 = 477 \text{ cycles.}$$

That is, in a submarine cable, the critical wave length  $U_0$  is relatively short, so that in long submarine cables, standing waves may appear, which are not oscillatory in time, but die out gradually. In such cables, due to their relative high resistance, the damping effect is very great,  $w = 1500$ , and standing waves rapidly die out.

In the investigation of the submarine cable, the complete equations must therefore be used, and  $q$  cannot always be assumed as large compared with  $m$  and  $w$ , except when dealing with local oscillations.

### 4. *Long Distance Overhead Telephone Circuit.*

14. Considering a telephone circuit of 1000 miles length, metallic return, consisting of two wires No. 4 B. & S. gauge, 24 in. distant from each other.

Per mile of conductor:

$$r = 1.31 \text{ ohms}$$

$$L = 1.84 \times 10^{-3} h$$

$$C = .0172 \times 10^{-6} f$$

as conductance  $g$  we may assume:

- (a)  $g = 0$ , that is, very perfect insulation, as in dry weather.
- (b)  $g = 2.5 \times 10^{-6}$ , that is, slightly leaky lines.
- (c)  $g = 12 \times 10^{-6}$ , that is, poor insulation, or a leaky line.
- (d)  $g = 40 \times 10^{-6}$ , that is, extremely poor insulation, as during heavy rain.

The condition may also be investigated where the line is loaded with inductance coils, spaced so close together, that in their effect we can consider this additional inductance as uniformly distributed. Let the total inductance, per unit length, be increased by the loading coils to:

$$L_1 = 9 \times 10^{-3} h$$

or about 5 times the normal value.

Denoting then the constants of the loaded line by the index 1, it is:

|                            | (a)                      | (b)                  | (c)                   | (d)                   |
|----------------------------|--------------------------|----------------------|-----------------------|-----------------------|
| $w$                        | 356                      | 429                  | 706                   | 1518                  |
| $w_1$                      | 73                       | 146                  | 423                   | 1236                  |
| $m$                        | 356                      | 283                  | 6                     | -806                  |
| $m_1$                      | 73                       | 0                    | -277                  | -1090                 |
| $\sigma = \sqrt{LC}$       | $= 5.63 \times 10^{-6}$  |                      |                       |                       |
| $\sigma_1 = \sqrt{L_1 C}$  | $= 12.45 \times 10^{-6}$ |                      |                       |                       |
| $k_0 = m\sqrt{LC}$         | $2 \times 10^{-3}$       | $1.6 \times 10^{-3}$ | $33.7 \times 10^{-6}$ | $4.56 \times 10^{-3}$ |
| $k_{01} = m_1\sqrt{L_1 C}$ | $91 \times 10^{-6}$      | 0                    | $3.45 \times 10^{-3}$ | $13.5 \times 10^{-3}$ |
| $U_0$                      | 3140                     | 3920                 | 187,000               | 1380                  |
| $U_{01}$                   | 69,000                   | $\infty$             | 1820                  | 464                   |
| $N_0$                      | 55.6                     | 45                   | .96                   | 128                   |
| $N_{01}$                   | 11.6                     | 0                    | 44                    | 173                   |

Hence, in a long distance telephone line, distributed leakage up to a certain amount, increases the critical wave length, and so makes even the long wave oscillatory. Beyond this amount, leakage again decreases the wave length. Distributed inductance, as by loading the line, increases the critical wave length, if the leakage is small, but in a badly leaky line, it decreases the critical wave length, and the amount of leakage, up to which an increase of the critical wave length occurs, is less, in a loaded line, that is, in a line of higher inductance.

In other words, a moderate amount of distributed leakage improves a long distance telephone line, an excessive amount of leakage spoils it. An increase of inductance, by loading the line, improves the line, if the leakage is small, but spoils the line, if the leakage is considerable. The amount of leakage up to which improvement in the telephone line occurs, is less in a loaded than in an unloaded line, that is, a loaded telephone line requires a far better insulation than an unloaded line.

#### IV. TRAVELING WAVES.

15. As seen, especially in electric power circuits, overhead or underground, the longest existing standing wave has a wave length which is so small compared with the critical wave length—where the frequency becomes zero—that the effect of the damping constant on frequency and wave length is negligible. The same obviously applies also to traveling waves, generally to a greater extent still, since traveling waves commonly are of wave lengths, which are only a small part of the length of the circuit. Usually therefore in the discussion of traveling waves, the effect of the damping constants on the frequency constant  $q$  and the wave length constant  $k$  can be neglected, that is, frequency and wave length assumed as independent of the energy loss in the circuit, and the equations (45) and (46) applied in dealing with the traveling wave.

In these equations, the distance traveled by the wave per second, is used as unit length, by the substitution:

$$\lambda = \sigma u$$

where:

$$\sigma = \sqrt{LC}$$

as this brings  $t$  and  $\lambda$  into direct comparison, and eliminates  $h$  and  $k$  from the equations, by the equation (43).

This gives, under the limiting conditions discussed before, the

*General Equations of the Traveling Wave:*

$$i = \epsilon^{-wt} \{ \epsilon^{+t(t-\lambda)} [B_1 \cos q(t-\lambda) + B_1' \sin q(t-\lambda)] - \epsilon^{+t(t+\lambda)} [B_2 \cos q(t+\lambda) + B_2' \sin q(t+\lambda)] + \epsilon^{-t(t-\lambda)} [B_3 \cos q(t-\lambda) + B_3' \sin q(t-\lambda)] - \epsilon^{-t(t+\lambda)} [B_4 \cos q(t+\lambda) + B_4' \sin q(t+\lambda)] \} \quad (61)$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ \epsilon^{+f(t-\lambda)} [B_1 \cos q(t-\lambda) + B_1' \sin q(t-\lambda)] + \epsilon^{+f(t+\lambda)} [B_2 \cos q(t+\lambda) + B_2' \sin q(t+\lambda)] + \epsilon^{-f(t-\lambda)} [B_3 \cos q(t-\lambda) + B_3' \sin q(t-\lambda)] + \epsilon^{-f(t+\lambda)} [B_4 \cos q(t+\lambda) + B_4' \sin q(t+\lambda)] \} \quad (62)$$

or:

$$e = \epsilon^{-wt} \{ \epsilon^{+f(t-\lambda)} [A_1 \cos q(t-\lambda) + A_1' \sin q(t-\lambda)] + \epsilon^{+f(t+\lambda)} [A_2 \cos q(t+\lambda) + A_2' \sin q(t+\lambda)] + \epsilon^{-f(t-\lambda)} [A_3 \cos q(t-\lambda) + A_3' \sin q(t-\lambda)] + \epsilon^{-f(t+\lambda)} [A_4 \cos q(t+\lambda) + A_4' \sin q(t+\lambda)] \} \quad (63)$$

$$i = \sqrt{\frac{C}{L}} \epsilon^{-wt} \{ \epsilon^{+f(t-\lambda)} [A_1 \cos q(t-\lambda) + A_1' \sin q(t-\lambda)] - \epsilon^{+f(t+\lambda)} [A_2 \cos q(t+\lambda) + A_2' \sin q(t+\lambda)] + \epsilon^{-f(t-\lambda)} [A_3 \cos q(t-\lambda) + A_3' \sin q(t-\lambda)] - \epsilon^{-f(t+\lambda)} [A_4 \cos q(t+\lambda) + A_4' \sin q(t+\lambda)] \} \quad (64)$$

where:

$$w = \frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right) \quad (65)$$

In these equations the values  $A$ ,  $B$ , are integration constants, which are determined by the terminal conditions of the problem.

The terms with  $(t-\lambda)$  may be considered as the main wave, the terms with  $(t+\lambda)$  as the reflected wave, or inversely, depending on the direction of propagation of the wave.

16. As the traveling wave consists of a main wave, with variable  $(t-\lambda)$ , and a reflected wave of the same character, but moving in opposite direction, thus with the variable  $(t+\lambda)$ , these waves may be studied separately, and afterwards the effect of their combination investigated.

Thus, considering at first one of the waves only, that with the variable  $(t-\lambda)$ , it is, from equations (63) to (64):

$$\begin{aligned} e &= \epsilon^{-wt} \{ \epsilon^{+f(t-\lambda)} [A_1 \cos q(t-\lambda) + A_1' \sin q(t-\lambda)] + \epsilon^{-f(t-\lambda)} [A_3 \cos q(t-\lambda) + A_3' \sin q(t-\lambda)] \} \\ &= \epsilon^{-wt} \{ (A_1 \epsilon^{+f(t-\lambda)} + A_3 \epsilon^{-f(t-\lambda)}) \cos q(t-\lambda) + (A_1' \epsilon^{+f(t-\lambda)} + A_3' \epsilon^{-f(t-\lambda)}) \sin q(t-\lambda) \} \end{aligned} \quad (66)$$



and:

$$i = \sqrt{\frac{C}{L}} e$$

That is, in a single traveling wave, current and voltage are in phase with each other, and proportional to each other, with an *effective impedance*:

$$z = \frac{e}{i} = \sqrt{\frac{L}{C}}$$

This proportionality between  $e$  and  $i$ , and coincidence of phase, obviously does not exist any more in the combination of main waves and reflected waves, since in reflection, the current reverses with the reversal of the direction of propagation, while the voltage remains in the same direction.

In equation (66), the time  $t$  appears only in the term  $(t - \lambda)$ , except in the factor  $\epsilon^{-wt}$ , while the distance  $\lambda$  appears only in the term  $(t - \lambda)$ . Substituting therefore:

$$\vartheta = t - \lambda$$

hence:

$$t = \vartheta + \lambda$$

that is, counting the time differently at any point  $\lambda$ , and counting it at every point of the circuit from the same point in the phase of the wave, from which the time  $t$  is counted at the starting point of the wave,  $\lambda = 0$ , or, in other words, shifting the starting point of the counting of time, with the distance  $\lambda$ , it is, substituted in (66):

$$\left. \begin{aligned} e &= \epsilon^{-wt} \{ \epsilon^{+f\vartheta} (A_1 \cos q \vartheta + A_1' \sin q \vartheta) + \epsilon^{-f\vartheta} (A_3 \cos q \vartheta \\ &\quad + A_3' \sin q \vartheta) \} \\ &= \epsilon^{-w\lambda} \epsilon^{-w\vartheta} \{ \epsilon^{+f\vartheta} (A_1 \cos q \vartheta + A_1' \sin q \vartheta) + \epsilon^{-f\vartheta} (A_3 \cos q \\ &\quad \vartheta + A_3' \sin q \vartheta) \} \\ &= \epsilon^{-w\lambda} \epsilon^{-w\vartheta} \{ (A_1 \epsilon^{+f\vartheta} + A_3 \epsilon^{-f\vartheta}) \cos q \vartheta + (A_1' \epsilon^{+f\vartheta} + A_3' \\ &\quad \epsilon^{-f\vartheta}) \sin q \vartheta \} \end{aligned} \right\} \quad (67)$$

The latter form of the equation is best suited to represent the variation of the wave, at a fixed point  $\lambda$  in space, as function of the local time  $\vartheta$ .

The wave is the product of a term:  $\epsilon^{-w}$  which decreases with increasing distance  $\lambda$ , and a term:

$$\begin{aligned} e_0 &= \epsilon^{-w\vartheta} \{ \epsilon^{+f\vartheta} (A_1 \cos q \vartheta + A_1' \sin q \vartheta) + \epsilon^{-f\vartheta} (A_2 \cos q \vartheta + A_2' \sin q \vartheta) \} \\ &= \epsilon^{-w\vartheta} \{ (A_1 \epsilon^{+f\vartheta} + A_2 \epsilon^{-f\vartheta}) \cos q \vartheta + (A_1' \epsilon^{+f\vartheta} + A_2' \epsilon^{-f\vartheta}) \sin q \vartheta \} \end{aligned} \quad (68)$$

which latter term is independent of the distance, but merely a function of the time  $\vartheta$ , when counting the time at any point of the line from the moment of the passage of the same phase of the wave.

Since the coefficient in the exponent of the distance decrement  $\epsilon^{-w\lambda}$ , contains only the circuit constants:

$$w = \frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right)$$

but does not contain  $f$  and  $q$ , or the other integration constants.

It follows herefrom:

The attenuation constant of a traveling wave, and therefore the distance decrement of the wave depends upon the circuit constants  $r$ ,  $L$ ,  $g$ ,  $C$ , only, but does not depend upon the wave length, frequency, voltage or current; hence all traveling waves in the same circuit die out at the same rate, regardless of their frequency and of their wave shape, or in other words, a complex traveling wave retains its wave shape, when traversing a circuit, and merely decreases in amplitude by the distance decrement  $\epsilon^{-w\lambda}$ .

The wave attenuation is a constant of the circuit.

This obviously applies only for waves of constant velocity, that is, such waves in which  $q$  is large compared with  $f$ ,  $w$ , and  $m$ , and does not strictly apply to extremely long waves, as discussed in 13.

17. By changing the line constants, as by inserting inductance in such a manner as to give the effect of uniform distribution (loading the line), the attenuation of the wave can be reduced, that is, the wave caused to travel a greater distance  $u$  with the same decrease of amplitude.

As function of the inductance  $L$ , the attenuation constant is a minimum for:

$$\frac{d w_0}{d L} = 0$$

hence:

$$r C - g L = 0$$

or:

$$\frac{L}{C} = \frac{r}{g}$$

and, if the conductance:

$$g = 0$$

it is:

$$L = \infty$$

Hence, in a perfectly insulated circuit, or circuit having no energy losses depending on the voltage, the attenuation decreases with increase of the inductance, that is, by "loading the line", and the more inductance is inserted, the better the telephonic transmission is.

In a leaky telephone line, increase of inductance decreases the attenuation, and so improves the telephonic transmission, up to the value of inductance,

$$L = \frac{r C}{g}$$

and beyond this value, inductance is harmful, by again increasing the attenuation.

For instance, if in a long distance telephone circuit per mile:

$$r = 1.31 \text{ ohms}$$

$$L = 1.84 \times 10^{-3} h$$

$$g = 1.0 \times 10^{-6} \text{ mho}$$

$$C = 0.0172 \times 10^{-6} f$$

the attenuation of a traveling wave or impulse is:

$$w_0 = 0.00217$$

hence, for a distance or length of line of  $l = 2000$  miles:

$$\begin{aligned}\epsilon^{-w_0 l} &= \epsilon^{-4.34} \\ &= 0.0129\end{aligned}$$

that is, the wave is reduced to 1.29 per cent. of its original value.

The best value of inductance is, by

$$L = \frac{r}{g} C = 0.0225 h$$

and in this case, the attenuation constant becomes:

$$w_0 = 0.00114$$

and thus:

$$\begin{aligned}\epsilon^{-w_0 l} &= \epsilon^{-2.24} \\ &= 0.1055\end{aligned}$$

or 10.55 per cent of the original value of the wave.

In this telephone circuit, by adding per mile an additional inductance of:

$$22.5 - 1.84 = 20.7 \text{ mh.}$$

the intensity of the arriving wave is increased from 1.29 per cent. to 10.55 per cent. or more than 8 times.

If however, in wet weather, the leakage increases to the value:

$$g = 5 \times 10^{-6}$$

in the unloaded line, it is:

$$\begin{aligned}w_0 &= 0.00282 \\ \epsilon^{-w_0 l} &= 0.0035\end{aligned}$$

while in the loaded line, it is:

$$\begin{aligned}w_0 &= 0.00341 \\ \epsilon^{-w_0 l} &= 0.0011\end{aligned}$$

and while with the unloaded line, the arriving wave is still 0.35 per cent. of the outgoing wave, in the loaded line, it is only 0.11 per cent.; that is, in this case, loading the line with in-

ductance has badly spoiled telephonic communication and increased the decay of the wave more than three-fold. A loaded telephone line is much more sensitive to changes of leakage  $g$ , that is, to meteorological conditions.

18. The equation of the traveling wave (67):

$$e = \epsilon^{-w\lambda} \epsilon^{-w\vartheta} \{ \epsilon^{+f\vartheta} (A_1 \cos q \vartheta + A_1 \sin q \vartheta) + \epsilon^{-f\vartheta} (A_3 \cos q \vartheta + A_3' \sin q \vartheta) \} \quad (67)$$

can be reduced to the form:

$$e = \epsilon^{-w\lambda} \{ E_1 \epsilon^{-w\vartheta_1} (\epsilon^{+f\vartheta_1} - \epsilon^{-f\vartheta_1}) \sin q \vartheta_1 + E_2 \epsilon^{-w\vartheta_2} (\epsilon^{+f\vartheta_2} - \epsilon^{-f\vartheta_2}) \cos q \vartheta_2 \} \quad (69)$$

where:

$$\vartheta_1 = \vartheta + \gamma_1 = t - \lambda - \gamma_1$$

$$\vartheta_2 = \vartheta + \gamma_2 = t - \lambda - \gamma_2$$

Any traveling wave can be resolved into, and considered as consisting of, a combination of 2 waves: the

*Traveling Sine Wave:*

$$e_1 = E_1 \epsilon^{-w\lambda} \epsilon^{-w\vartheta} (\epsilon^{+f\vartheta_1} - \epsilon^{-f\vartheta_1}) \sin q \vartheta_1 \quad (70)$$

*Traveling Cosine Wave:*

$$e_2 = E_2 \epsilon^{-w\lambda} \epsilon^{-w\vartheta_2} (\epsilon^{+f\vartheta_2} - \epsilon^{-f\vartheta_2}) \cos q \vartheta_2 \quad (71)$$

Since  $q$  is a large quantity compared with  $w$  and  $f$ , the two component traveling waves, (70) and (71), differ appreciably from each other in appearance only for very small values of  $\vartheta$ , that is, near  $\vartheta_1 = 0$  and  $\vartheta_2 = 0$ . The traveling sine wave rises in the first half cycle very slightly, while the traveling cosine wave rises rapidly, *i.e.*, the tangent of the angle which

the wave makes with the horizontal, or  $\frac{de}{dt}$ , equals 0 with the

sine wave, and has a definite value with the cosine wave.

All traveling waves in an electric circuit can be resolved into constituent elements, traveling sine waves and traveling cosine waves, and the general traveling wave consists of four component waves, a sine wave, its reflected wave, a cosine wave, and its reflected wave.

The elements of the traveling wave, the traveling sine wave  $e_1$ , and the traveling cosine wave  $e_2$ , contain 4 constants:

The intensity constant:  $E$

The attenuation constant:  $w$  respectively  $w_0$

The frequency constant:  $f$

And the constant:  $\vartheta$

The wave starts from zero, builds up to a maximum, and then gradually dies out, to zero at infinite time.

The absolute term of the wave, that is, the term which represents the values between which the wave oscillates, is:

$$e_0 = E \epsilon^{-w\lambda} \epsilon^{-w\vartheta} (\epsilon^{+j\vartheta} - \epsilon^{-j\vartheta}) \quad (72)$$

This term  $e_0$  may be called the *amplitude* of the wave. The amplitude  $e_0$  is a maximum for the value of  $\vartheta$ , given by:

$$\frac{d e_0}{d \vartheta} = 0$$

$$\vartheta_0 = \frac{1}{2f} \log \frac{w+f}{w-f} \quad (73)$$

and, substituting this value into the equation of the absolute term of the wave, (72), gives:

$$e_m = E \epsilon^{-w\lambda} \frac{2f}{\sqrt{w^2 - f^2}} \left( \frac{w+f}{w-f} \right)^{-\frac{w}{2f}}$$

The rate of building up of the wave, or the steepness of the wave front, is given by:

$$\begin{aligned} G_0 &= \frac{d e_0}{d \vartheta} \bigg|_{\vartheta=0} \\ &= 2f E \epsilon^{-w\lambda} \end{aligned}$$

It is a maximum, that is,  $\vartheta_0$  a minimum, for the value of  $f$ , in equation (73), given by:

$$\frac{d \vartheta_0}{d f} = 0$$

or:

$$f = 0$$

or the standing wave, which rises infinitely fast, that is, appears instantly.

The smaller therefore  $f$  is, the more rapidly is the rise of the traveling wave, and  $f$  may be called the *acceleration constant* of the traveling wave.

$\vartheta$  is the time counted from the beginning of the wave.

Since:

$$\vartheta = t - \lambda - \gamma$$

if we change the zero point of the distance, that is, count the distance  $\lambda$  from that point of the line, at which the wave starts at time  $t = 0$ , or in other words, count time  $t$  and distance  $\lambda$  from the origin of the wave, it is:

$$\vartheta = t - \gamma$$

and the traveling wave so may be represented by:

The amplitude:

$$e_0 = E \epsilon^{-w t} (\epsilon^{+t \vartheta} - \epsilon^{-t \vartheta})$$

The sine wave:

$$e_1 = E \epsilon^{-w t} (\epsilon^{+t \vartheta} - \epsilon^{-t \vartheta}) \sin q \vartheta = e_0 \sin q \vartheta \quad (74)$$

The cosine wave:

$$e_2 = E \epsilon^{-w t} (\epsilon^{+t \vartheta} - \epsilon^{-t \vartheta}) \cos q \vartheta = e_0 \cos q \vartheta$$

and  $\vartheta = t - \lambda$  can be considered as the distance, counting backwards from the wave front, or the distance counted with the point  $\lambda$ , which the wave has just reached, as zero point, and in opposite direction to  $\lambda$ , or as the time, counting from the moment of the arrival of the wave front.

Equation (74) represents the distribution of the wave along the line, at the moment  $t$ , or the variation of the wave with the time, at a point  $\lambda$ .

As seen, the wave maintains its shape, but progresses along the line, and at the same time dies out, by the time decrement  $\epsilon^{-w t}$ .

19. As an instance may be considered a traveling wave of the constants:

$$w = 115$$

$$f = 45$$

$$q = 2620$$

$$E = 100$$



hence:

$$\begin{aligned} e_0 &= 100 \epsilon^{-115t} (\epsilon^{+45\vartheta} - \epsilon^{-45\vartheta}) \\ &= 100 \epsilon^{-115\lambda} (\epsilon^{-70\vartheta} - \epsilon^{-160\vartheta}) \end{aligned}$$

where:

$$\vartheta = t - \lambda$$

In Fig. 2 is shown the amplitude  $e_0$ , as function of the distance  $\lambda$ , for the different values of time:

$$t = 2, 4, 8, 12, 16, 20, 24 \text{ and } 32 \times 10^{-3}$$

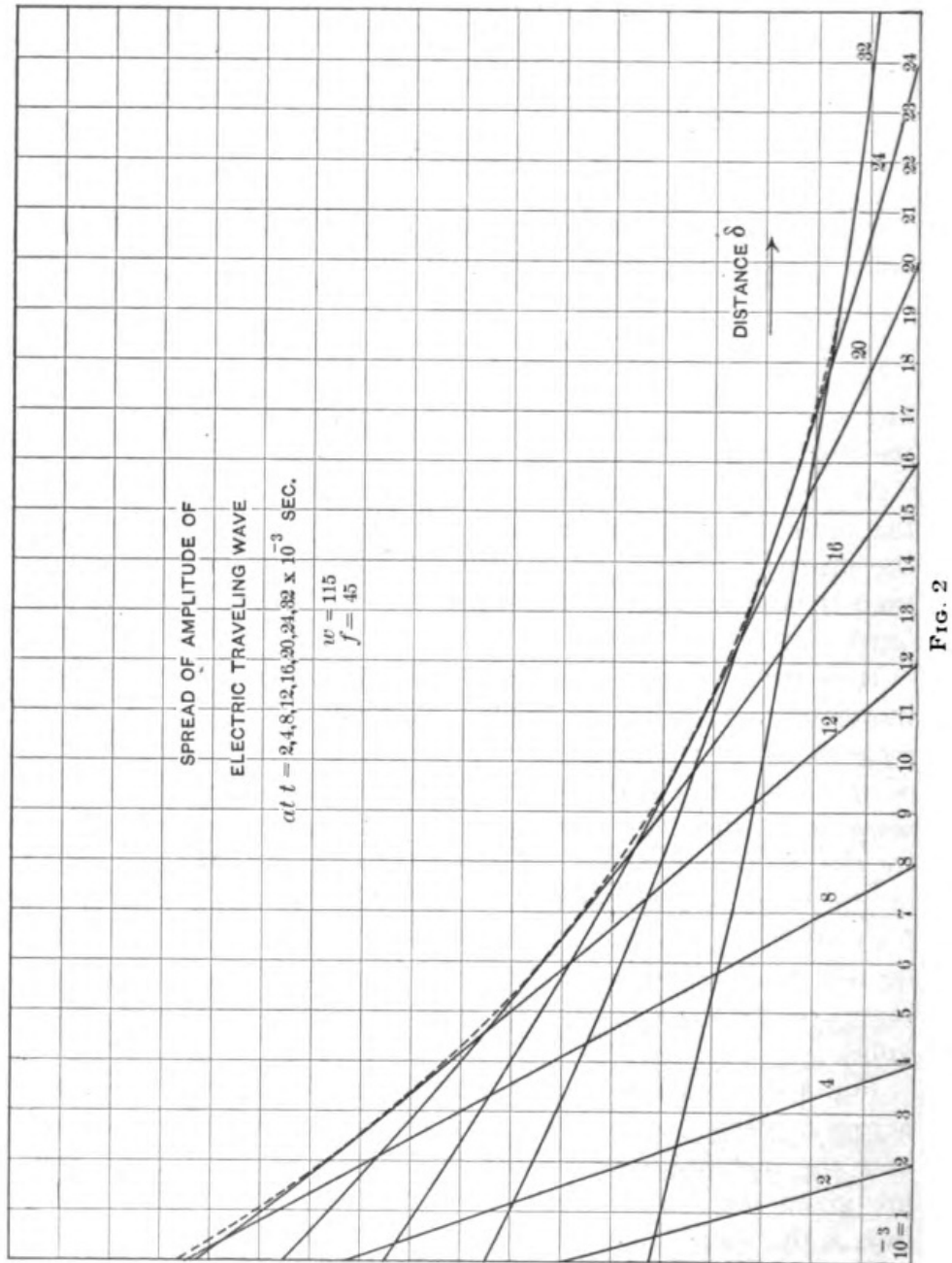
with the maximum amplitude  $e_m$ , in dotted line, as envelope of the curves of  $e_0$ .

As seen, the amplitude of the wave gradually rises, and at the same time spreads over the line, reaching the maximum at the starting point  $\lambda = 0$  at the time  $t_0 = 9.2 \times 10^{-3}$  sec., and then decreases again while continuing to spread over the line, until it gradually dies out.

It is interesting to note that the distribution curves of the amplitude are nearly straight lines, but also, that even in the longest power transmission line the wave has reached the end of the line, and reflection occurs, before the maximum of the wave is reached: the unit of length  $\lambda$  is the distance traveled by the wave per second, or 188,000 miles, and during the rise of the wave, at the origin, from its start to the maximum, or  $9.2 \times 10^{-3}$  sec., the wave has traveled 1760 miles, thus the reflected wave would have returned to the origin before the maximum of the wave is reached, if the circuit is shorter than 880 miles.

Fig. 3 shows the passage of the traveling wave:  $e_1 = e_0 \sin q \vartheta$ , across a point  $\lambda$  of the line, with the local time  $\vartheta$  as abscissas, and the instantaneous values of  $e_1$  as ordinates. The values are given for  $\lambda = 0$ , where  $\vartheta = t$ ; for any other point of the line,  $\lambda$ , the wave shape is the same, but all the ordinates reduced by the factor  $\epsilon^{-115\lambda}$ , in the proportion as shown in the dotted curve in Fig. 2.

Fig. 4 shows the beginning of the passage of the traveling wave across a point  $\lambda=0$  of the line, that is, the starting of a



wave, or its first one and one-half cycles, for the trigonometric functions differing successively by 45 degrees, that is,

$$e_1 = e_0 \sin q \vartheta$$

$$\frac{e_1 + e_2}{\sqrt{2}} = e_0 \sin \left( q \vartheta + \frac{\pi}{4} \right)$$

$$e_2 = e_0 \cos q \vartheta = e_0 \sin \left( q \vartheta + \frac{\pi}{2} \right)$$

$$\frac{e_2 - e_1}{\sqrt{2}} = e_0 \cos \left( q \vartheta + \frac{\pi}{4} \right) = e_0 \sin \left( q \vartheta + \frac{3\pi}{4} \right)$$

the first curve of Fig. 4 so is the beginning of Fig. 3.

In waves traveling over a water surface, shapes like Fig. 4 can be observed.

For the purpose of illustration, however, in Figs. 3 and 4 the oscillations are shown far longer than usually occur: the value:  $q = 2620$ , corresponds to a frequency:  $N = 418$  cycles, while traveling waves are more common of frequencies from 100 to 10,000 times as high.

20. A specially interesting traveling wave is the wave in which:

$$f = w$$

since in this wave the time decrement of the first main wave and its reflected wave vanishes:

$$e^{-(w-f)t} = 0$$

that is, the first main wave and its reflected wave are not transient, but permanent, that is, alternating waves, and the equations of the first main wave give the equation of the alternating-current circuit with distributed  $r$ ,  $L$ ,  $g$ ,  $C$ , which appears as a special case of a traveling wave.

Since in this case, the frequency, and so the value of  $q$  is low, and comparable with  $w$  and  $f$ , the approximations made in the previous discussion of the traveling wave are not permissible, but the general equations have to be used.

Substituting therefore in (29) and (30):

$$f = w$$

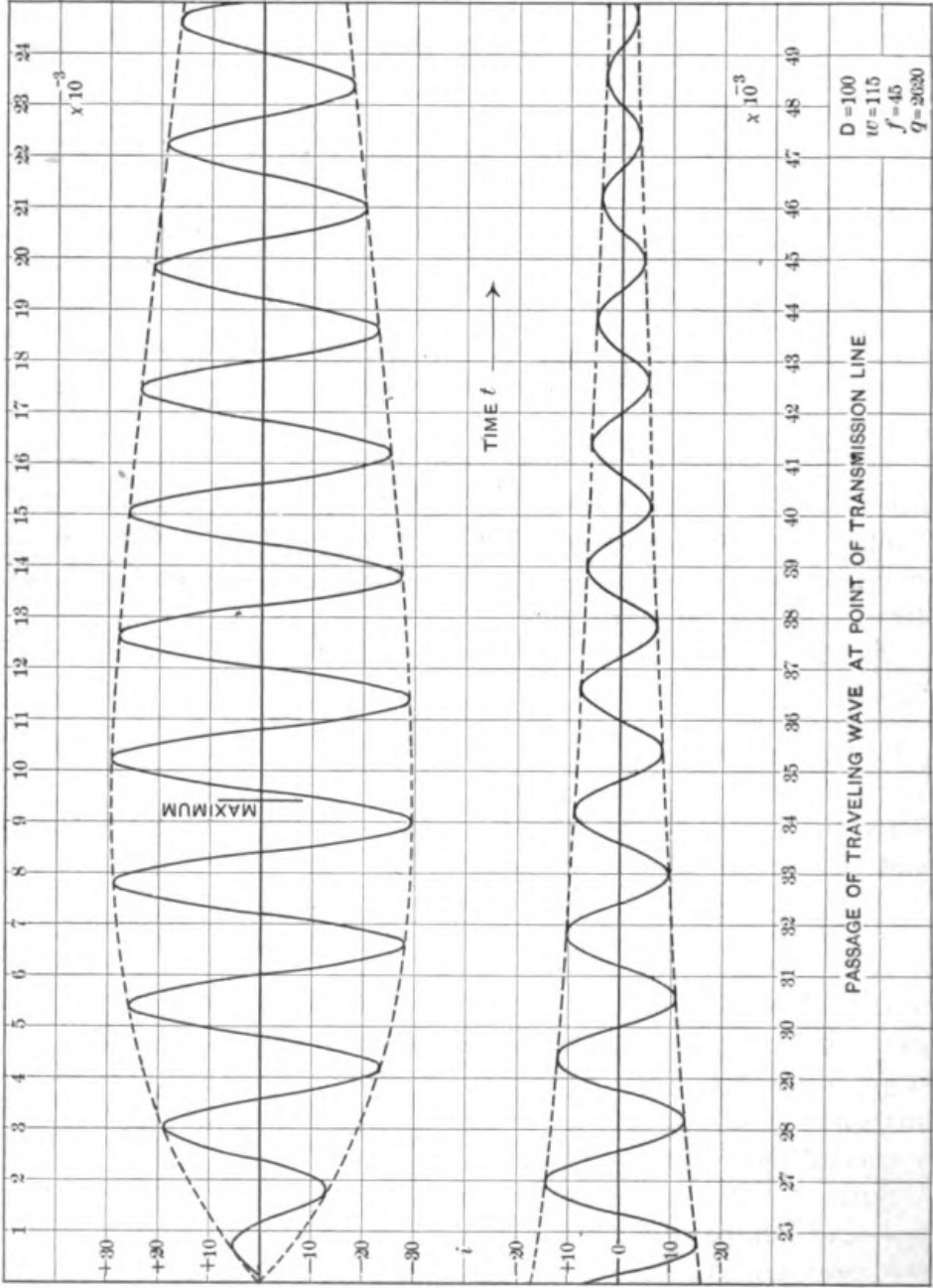


FIG. 3

gives:

$$i = [\epsilon^{-hu} \{C_1 \cos (qt - ku) + C_1' \sin (qt - ku)\} - \epsilon^{+hu} \{C_2 \cos (qt + ku) + C_2' \sin (qt + ku)\}] - \epsilon^{-2wt} [\epsilon^{-hu} \{C_4 \cos (qt + ku) + C_4' \sin (qt + ku)\} - \epsilon^{+hu} \{C_3 \cos (qt - ku) + C_3' \sin (qt - ku)\}] \quad (75)$$

and:

$$e = [\epsilon^{-hu} \{(c_1' C_1' - c_1 C_1) \cos (qt - ku) - (c_1' C_1 + c_1 C_1') \sin (qt - ku)\} + \epsilon^{+hu} \{(c_1' C_2' - c_1 C_2) \cos (qt + ku) - (c_1' C_2 + c_1 C_2') \sin (qt + ku)\}] + \epsilon^{-2wt} [\epsilon^{-hu} \{(c_2' C_4' - c_2 C_4) \cos (qt + ku) - (c_2' C_4 + c_2 C_4') \sin (qt + ku)\} + \epsilon^{+hu} \{(c_2' C_3' - c_2 C_3) \cos (qt - ku) - (c_2' C_3 + c_2 C_3') \sin (qt - ku)\}] \quad (76)$$

In these equations of current  $i$  and voltage  $e$ , the first term represents the usual equations of the distribution of alternating current and voltage in a long distance transmission line, and can by the substitution of complex quantities be reduced to the form usually given.

The second term is a transient term of the same frequency. That is:

In a long distance transmission line or other circuits of distributed  $r$ ,  $L$ ,  $g$ ,  $C$ , when traversed by alternating current under an alternating impressed voltage, at a change of circuit conditions a transient term of fundamental frequency may appear, which has the time decrement, that is, dies out at the rate:

$$\epsilon^{-2wt} = \epsilon^{-\left(\frac{r}{L} + \frac{g}{C}\right)t}$$

It is interesting to note, that here the general equations of alternating-current long distance transmission appear as a special case of the equation of the traveling wave, and indeed can be considered as a section of a traveling wave, in which the acceleration constant  $f$  equals the exponential decrement  $w$ , and the attenuating constant  $\alpha$  of the usual transmission line equation is the constant  $h$ ; the wave length constant  $\beta$  is the constant  $k$ .

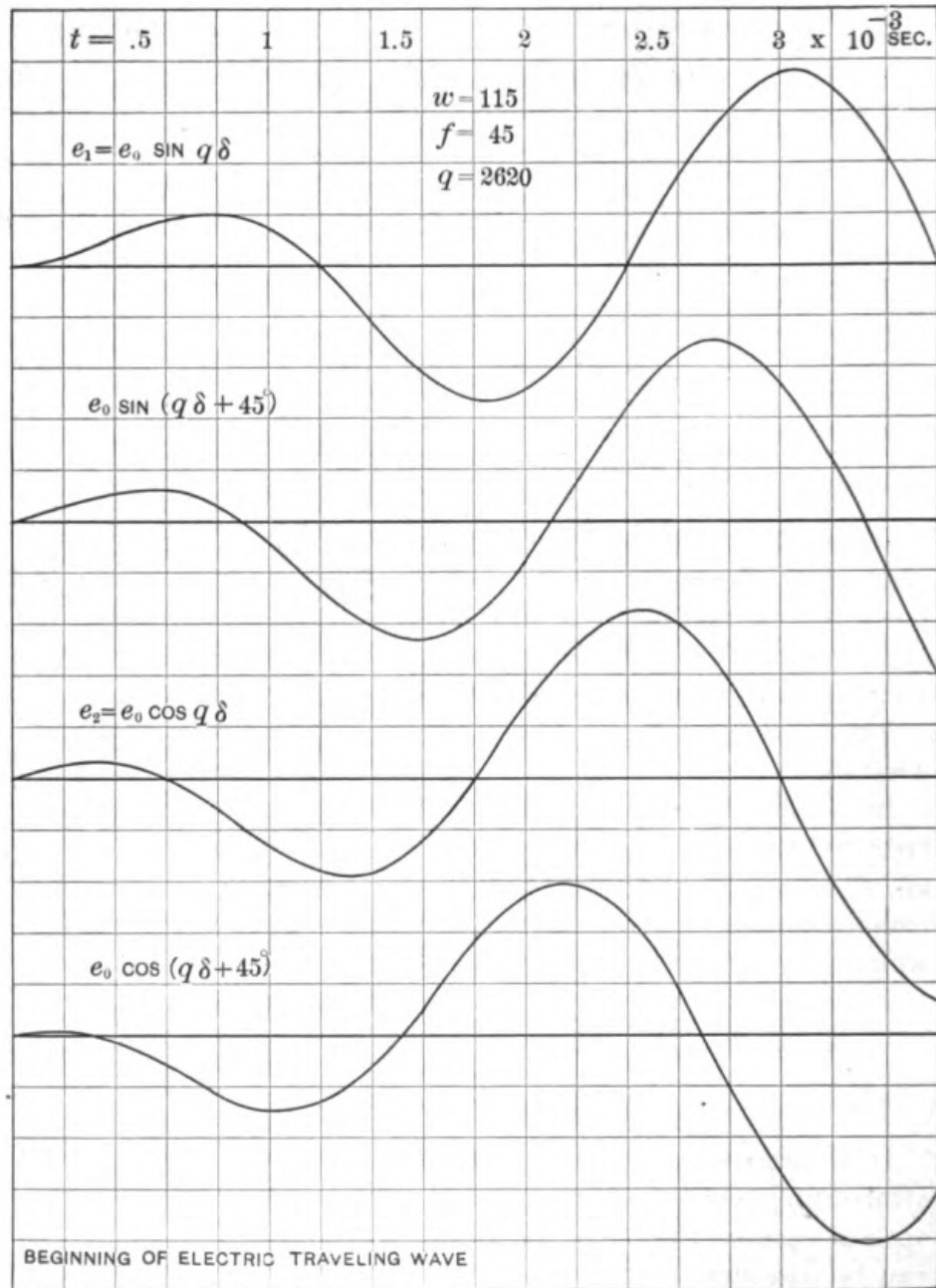


FIG. 4



## V. FREE OSCILLATIONS.

21. The general equations of the electric circuit (29) and (30), contain 8 terms: 4 waves, 2 main waves and their reflected waves, and each wave consists of a sine term and a cosine term.

The equations contain 5 constants:

The frequency constant  $q$

The wave length constant  $k$

The time attenuation constant  $w$

The distance attenuation constant  $k$

The time acceleration constant  $f$

of which the time attenuation  $w$  is a constant of the circuit, independent of the character of the wave.

By the value of the acceleration constant  $f$ , waves may be subdivided into:

$f = 0$ : standing waves,

$w > f > 0$ : traveling waves,

$f = w$ : alternating, current and voltage waves.

The general equations contain 8 integration constants  $C$  and  $C'$ , which have to be determined by the terminal conditions of the problem.

Upon the values of these integration constants  $C$  and  $C'$  largely depends the difference between the phenomena occurring in electric circuits, as those due to direct currents or pulsating currents, alternating currents, oscillating currents, inductive discharges, etc., and the study of the *terminal conditions* is of the foremost importance.

*Free oscillations* are understood as the transient phenomena occurring in an electric circuit or part of the circuit, to which neither electric energy is supplied by some outside source, nor electric energy abstracted therefrom.

Free oscillations thus are the transient phenomena resulting from the dissipation of the energy stored in the electric field of the circuit, or inversely, the accumulation of the energy of the electric field, and their appearance so presupposes the possibility of energy storage in more than one form, so as to allow an interchange or surge of energy between its different forms, electromagnetic and electrostatic energy. Free oscillations occur only in circuits containing capacity  $C$  as well as inductance  $L$ .

The absence of energy supply or obstruction so defines the free oscillations by the conditions, that the power  $p = e i$  at the two ends of the circuit or section of the circuit must be zero at all times, or the circuit must be closed upon itself.

With a circuit of uniformly distributed constants, the latter case is not realized, but is of foremost importance in a complex circuit, that is, a circuit comprising sections of different constants.

Representing then the two ends of the electric circuit by  $u = 0$  and  $u = l$ , that is, counting the distance from one end of the circuit, the condition of a free oscillation is:

$$\begin{array}{ll} u = 0: & p = 0: \\ u = l: & p = 0. \end{array}$$

since  $p = e i$ , this means that at  $u = 0$  and  $u = l$ , either  $e$  or  $i$  must be zero.

This gives 4 sets of terminal conditions:

- |                         |                      |
|-------------------------|----------------------|
| 1. $e = 0$ at $u = 0$ ; | $i = 0$ at $u = l$ . |
| 2. $i = 0$ at $u = 0$ ; | $e = 0$ at $u = l$ . |
| 3. $e = 0$ at $u = 0$ ; | $e = 0$ at $u = l$ . |
| 4. $i = 0$ at $u = 0$ ; | $i = 0$ at $u = l$ . |

Case 2 represents the same condition as 1, merely with the distance  $u$  counting from the other end of the circuit: a line open at one end, grounded at the other end. Case 3 represents a circuit open at both ends, and Case 4 a circuit grounded at both ends.

Substituting these terminal conditions into the general equations it follows:

$$h = 0$$

that is, the *free oscillation of a circuit is a standing wave*.

And also:

$$k l = \frac{(2n+1)\pi}{2} \quad (77)$$

if  $e = 0$  at one,  $i = 0$  at the other end of the circuit, and

$$k l = n\pi \quad (78)$$

if either  $e = 0$  at both ends of the circuit, or  $i = 0$  at both ends of the circuit.



From (77) it follows, that

$$k l = \frac{\pi}{2},$$

or an *odd* multiple thereof. That is, the longest wave which can exist in the circuit, is that which makes the circuit a quarter wave length. Besides this fundamental wave, all its odd multiples can exist. Such an oscillation may be called a *quarter wave oscillation*.

The oscillation of a circuit, which is open at one end, grounded at the other end, is a quarter wave oscillation, which can contain only the odd harmonics of the fundamental wave of oscillation.

From (78) it follows, that

$$k l = \pi$$

or a multiple thereof. That is, the longest wave, which can exist in such a circuit, is that wave, which makes the circuit a half wave length. Besides this fundamental wave, all its multiples, odd as well as even, can exist. Such an oscillation may be called a *half wave oscillation*.

The oscillations of a circuit which is open at both ends, or grounded at both ends, is a half wave oscillation, and a half wave oscillation can also contain the even harmonics of the fundamental wave of oscillation, and so also the constant term, for  $n = 0$  in (78).

It is interesting to note, that in the half wave oscillation of a circuit, we have a case of a circuit, in which *even* higher harmonics exist, and the electromotive force and current wave are not symmetrical.

22. From  $h = 0$  follows, by equation:

$$f = 0, \quad \text{if: } k^2 > L C m^2,$$

and:

$$q = 0, \quad \text{if: } k^2 < L C m^2,$$

The smallest value of  $k$ , which can exist, is, by equation (77):

$$k = \frac{\pi}{2l}$$

and, as discussed in paragraph (15), this value in high potential high power circuits usually is very much larger than  $LCm^2$ , so that the case:  $q = 0$ , is realized only in extremely long circuits, as long distance telephone or submarine cable, but not in transmission lines, and, the first case:  $f = 0$ , is mainly of importance.

Substituting therefore:  $h = 0$  and  $f = 0$ , dropping terms of higher order, and rearranging, finally gives the

*Equations of the Free Oscillation:*

$$\left. \begin{aligned} i &= \Sigma A \epsilon^{-wt} \{ \cos (qt - ku - \gamma) \pm \cos (qt + ku - \gamma) \} \\ e &= \sqrt{\frac{L}{C}} \Sigma A \epsilon^{-wt} \{ \cos (qt - ku - \gamma) \mp \cos (qt + ku - \gamma) \} \end{aligned} \right\} \quad (79)$$

While the free oscillation of a circuit is a standing wave, the general standing wave, as represented by equations (59) and (60), with 4 integration constants  $A_1, A_1', A_2, A_2'$ , is not necessarily a free oscillation.

To be a free oscillation, the power  $e i$ , that is, either  $e$  or  $i$ , must be zero at two points of the circuit, the ends of the circuit or section of circuit, which oscillates.

This gives:

$$A_1^2 + A_1'^2 = A_2^2 + A_2'^2$$

as the condition, which must be fulfilled between the integration constants, to make a standing wave a free oscillation, and as can easily be shown.

If the integration constants of a standing wave fulfil the conditions:

$$A_1^2 + A_1'^2 = A_2^2 + A_2'^2 = B^2$$

the circuit of this wave contains points  $u_1$ , distant from each other by half a wave length, at which  $e = 0$ , and points  $u_2$ , distant from each other by half a wave length, at which  $i = 0$ , and the points  $u_2$  are intermediate between the points  $u_1$ , that is, distant therefrom by one-quarter wave length. Any section of the circuit, from a point  $u_1$  or  $u_2$  to any other point  $u_1$  or  $u_2$ , then is a free oscillating circuit.

In the free oscillation of the circuit, either the circuit is bounded by one point  $u_1$  and one point  $u_2$ , that is, the voltage is zero at one end, and the current zero at the other end of the circuit,

and the circuit then is a quarter wave or an odd multiple thereof, or the circuit is bounded by two points  $u_1$ , or by two points  $u_2$ , and then the voltage is zero at both ends of the circuit, or the current is zero at both ends of the circuit, and in either case the circuit is one half wave or a multiple thereof.

23. Choosing one of the points  $u_1$  or  $u_2$  as starting point of the distance, and introducing again the velocity of propagation as unit distance:

$$\lambda = \sigma u$$

$$\sigma = \sqrt{LC}$$

and denoting:

$$\left. \begin{aligned} \phi &= \frac{2\pi}{\lambda_0} t \\ \omega &= \frac{2\pi}{\lambda_0} \lambda = \frac{2\pi \sqrt{LC}}{\lambda_0} u \end{aligned} \right\} \quad (80)$$

That is, representing a complete cycle of the fundamental frequency, or complete wave in time, by  $\phi = 2\pi$ , and a complete wave in space, by  $\omega = 2\pi$ .

#### A. Quarter Wave Oscillation:

(a)  $e = 0$  at  $u = 0$  (or:  $\omega = 0$ ):

$$\left. \begin{aligned} e &= \varepsilon^{-\omega t} \sum_{n=0}^{\infty} B_n \sin (2n+1) \omega \sin [(2n+1) \phi + \gamma_n] \\ i &= \sqrt{\frac{C}{L}} \varepsilon^{-\omega t} \sum_{n=0}^{\infty} B_n \cos (2n+1) \omega \cos [(2n+1) \phi + \gamma_n] \end{aligned} \right\} \quad (81)$$

(b)  $i = 0$  at  $u = 0$  (or:  $\omega = 0$ ):

$$\left. \begin{aligned} e &= \varepsilon^{-\omega t} \sum_{n=0}^{\infty} B_n \cos (2n+1) \omega \cos [(2n+1) \phi + \gamma_n] \\ i &= \sqrt{\frac{C}{L}} \varepsilon^{-\omega t} \sum_{n=0}^{\infty} B_n \sin (2n+1) \omega \sin [(2n+1) \phi + \gamma_n] \end{aligned} \right\} \quad (82)$$

B. *Half-Wave Oscillation:*(a)  $e = 0$  at  $u = 0$  (or:  $\omega = 0$ ):

$$\left. \begin{aligned} e &= \epsilon^{-wt} \sum_0^{\infty} B_n \sin n \omega \sin (n \phi + \gamma_n) \\ i &= \sqrt{\frac{C}{L}} \epsilon^{-wt} \sum_0^{\infty} B_n \cos n \omega \cos (n \phi + \gamma_n) \end{aligned} \right\} \quad (83)$$

(b)  $i = 0$  at  $u = 0$  (or:  $\omega = 0$ ):

$$\left. \begin{aligned} e &= \epsilon^{-wt} \sum_0^{\infty} B_n \cos n \omega \cos (n \phi + \gamma_n) \\ i &= \sqrt{\frac{C}{L}} \epsilon^{-wt} \sum_0^{\infty} B_n \sin n \omega \sin (n \phi + \gamma_n) \end{aligned} \right\} \quad (84)$$

where:

$$\left. \begin{aligned} \phi &= \frac{2\pi}{\lambda_0} t \\ \omega &= \frac{2\pi}{\lambda_0} \sqrt{LC} u \end{aligned} \right\} \quad (85)$$

$$\left. \begin{aligned} \lambda_0 &= 4l\sqrt{LC} \text{ in a quarter-wave,} \\ &= 2l\sqrt{LC} \text{ in a half-wave oscillation.} \end{aligned} \right\} \quad (86)$$

and:

$$\epsilon^{-wt} = \epsilon^{-\frac{\lambda_0 w}{2\pi} \phi} \quad \} \quad (87)$$

$\lambda_0$  is the wave length, and thus  $l/\lambda_0$  the frequency, of the fundamental wave, with the velocity of propagation as distance unit.

It is interesting to note, that the time decrement of the free oscillation,  $\epsilon^{-wt}$ , is the same for all frequencies and wave lengths, and that the relative intensity of the different harmonic components of the oscillation, and thereby the wave shape of the oscillation, remains unchanged during the decay of the oscillation.

This result, analogous to that found in the discussion on traveling waves, obviously is based on the assumption, that the

constants of the circuit do not change with the frequency. This, however, is not perfectly true; at very high frequencies,  $r$  increases, due to unequal current distribution in the conductor, and  $L$  slightly decreases hereby.  $g$  increases by the energy losses resulting from brush discharges and from electrostatic radiation, etc., so that in general, at very high frequency, an increase of  $r/L$  and  $g/C$ , and so of  $w$  may be expected, that is, a greater rapidity of the dying out of very high harmonics. This would result in smoothing out the wave shape with increasing decay, that is, making it more approach the fundamental and its lower harmonics.

24. The equations of a free oscillation of a circuit, as quarter wave or half wave, still contain the pairs of integration constants  $B_n$  and  $\gamma_n$ , representing respectively the intensity and the phase of the  $n$ th harmonic.

These pairs of integration constants are determined by the terminal conditions of time, that is, they depend upon the amount and the distribution of the stored energy of the circuit at the starting moment of the oscillation. or in other words, on the distribution of current and voltage at  $t = 0$ .

The voltage  $e_0$  and current  $i_0$ , at time  $t = 0$ , can be expressed as an infinite series of trigonometric functions of the distance  $u$ , that is, the distance angle  $\omega$ , or a Fourier series of such character as also to fulfil the terminal conditions in space, as discussed above, that is,  $e = 0$  respectively  $i = 0$  at the ends of the circuit.

The voltage and current distribution in the circuit, at the starting moment of the oscillation,  $t = 0$  respectively  $\phi = 0$ , so determines the integration constants in the usual manner.

25. As an instance the discharge of a transmission line may be considered:

A circuit of length  $l$  is charged to a uniform voltage  $E$ , while no current passes in the circuit. This circuit then is grounded at one end, while the other end remains insulated.

Let the distance be counted from the grounded end, and the time from the moment of grounding.

The terminal conditions then are:

$$\begin{array}{ll} \text{(a)} & \omega = 0: & e = 0 \\ & \omega = \pi/2: & i = 0 \end{array}$$

(b) at  $\phi = 0$ :

$$e = 0 \text{ for: } \omega = 0; e = E \text{ for: } \omega \pm 0$$

$$i = 0 \text{ for: } \omega \pm 0; i = \text{indefinite for: } \omega = 0.$$

The distribution of voltage  $e_0$  and current  $i_0$  in the circuit, at the starting moment  $\phi = 0$ , can be expressed by the Fourier series, and it is,

$$e = \frac{4}{\pi} \frac{E}{\epsilon^{-\omega t}} \sum_{n=1}^{\infty} \frac{\sin(2n+1)\omega \cos(2n+1)\phi}{2n+1}$$

$$i = \frac{4}{\pi} \frac{E}{L} \sqrt{\frac{C}{L}} \epsilon^{-\omega t} \sum_{n=1}^{\infty} \frac{\cos(2n+1)\omega \sin(2n+1)\phi}{2n+1}$$

Choosing the constants:

$$\begin{aligned} l &= 120 \text{ miles} \\ r &= 0.41 \text{ ohms} \\ L &= 1.95 \times 10^{-3} h \\ g &= 0.25 \times 10^{-6} \text{ mho} \\ C &= 0.0162 \times 10^{-3} f \\ E &= 40,000 \text{ volts.} \end{aligned}$$

$$e = 51,000 \epsilon^{-0.0485\phi} \left\{ \sin \omega \cos \phi + \frac{1}{3} \sin 3\omega \cos 3\phi + \frac{1}{5} \sin 5\omega \cos 5\phi + \dots \right\} \text{ volts}$$

$$i = 147.2 \epsilon^{-0.485\phi} \left\{ \cos \omega \sin \phi + \frac{1}{3} \cos 3\omega \sin 3\phi + \frac{1}{5} \cos 5\omega \sin 5\phi + \dots \right\} \text{ amperes.}$$

## VI. TRANSITION POINTS AND THE COMPLEX CIRCUIT.

26. The discussions of standing waves and free oscillations, and traveling waves, in the preceding, directly apply only to simple circuits, that is, circuits comprising a conductor of uniformly distributed constants  $r, L, g, C$ . Industrial electric circuits, however, never are simple circuits, but are always complex circuits comprising sections of different constants, generator, transformer, transmission lines and load; and a simple circuit is realized only by a section of a circuit, as a transmission line or a high potential transformer coil, which is cut off at both ends from the rest of the circuit, either by open circuiting:  $i = 0$ , or by short circuiting:  $e = 0$ . Approximately, the simple circuit is realized by a section of a complex circuit, connecting to other sections of very different constants, so that the ends of the circuit can, approximately, be considered as reflection points. For instance, an underground cable, of low  $L$  and high  $C$ , when connected to a large reactive coil, of high  $L$  and low  $C$ , may, approximately, at its ends be considered as having reflection

tion points:  $i = 0$ . A high potential transformer coil, of high  $L$  and low  $C$ , when connected to a cable, of low  $L$  and high  $C$ , may at its ends be considered as having reflection points:  $e = 0$ . That is, in other words, in the first case the reactive coil may be considered as stopping the flow of current, in the latter case the cable considered as short circuiting the transformer. This approximation however, while frequently relied upon in engineering practice, and often permissible for the circuit section in which the transient phenomenon originates, is not permissible in considering the effect of the phenomenon on the adjacent sections of the circuit. For instance, in the first case above mentioned, a transient phenomenon in an underground cable connected to a high reactance: the current and voltage in the cable may approximately be represented by considering the reactive coil as a reflection point, that is, an open circuit, since only a small current enters the reactive coil. Such a small current, entering the reactive coil, may however, give a very high and destructive voltage in the reactive coil, due to its high  $L$ , and so in the circuit beyond the reactive coil. In the investigation of the effect of a transient phenomenon originating in one section of a complex circuit—as an oscillating arc on an underground cable—on other sections of the circuit—as the generating station—even a very great change of circuit constants, cannot be considered as a reflection point. Since this is the most important case met in industrial practice, as disturbances originating in one section of a complex circuit usually develop their destructive effects in other sections of the circuit, the investigation of the general problem of a complex circuit comprising sections of different constants becomes necessary. This requires the investigation of the changes occurring in an electric wave, and its equations, when passing over a transition point from one circuit or section of a circuit into another section of different constants.

The equations (29) and (30), while most general, are less convenient for studying the transition of a wave from one circuit to another circuit, of different constants, and since in industrial high voltage circuits at least for waves originating in the circuits,  $q$  and  $k$  are very large compared with  $f$  and  $h$ , as discussed before,  $f$  and  $h$  may be neglected compared with  $q$  and  $k$ . This gives the equations (45) and (46), or, substituting (47) and reversing the sign of  $\lambda$ , that is, counting the distance in opposite direction:



$$\begin{aligned}
i = \epsilon^{-wt} \{ & \epsilon^{-f(\lambda-t)} [C_1 \cos q(\lambda-t) + C_1' \sin q(\lambda-t)] \\
& - \epsilon^{+f(\lambda+t)} [C_2 \cos q(\lambda+t) + C_2' \sin q(\lambda+t)] \\
& + \epsilon^{+f(\lambda-t)} [C_3 \cos q(\lambda-t) + C_3' \sin q(\lambda-t)] \\
& - \epsilon^{-f(\lambda+t)} [C_4 \cos q(\lambda+t) + C_4' \sin q(\lambda+t)] \} \quad (88)
\end{aligned}$$

where the constants are given by equations (37) to (43).

$$\begin{aligned}
e = \sqrt{\frac{L}{C}} \epsilon^{-wt} \{ & \epsilon^{-f(\lambda-t)} [C_1 \cos q(\lambda-t) + C_1' \sin q(\lambda-t)] \\
& + \epsilon^{+f(\lambda+t)} [C_2 \cos q(\lambda+t) + C_2' \sin q(\lambda+t)] \\
& + \epsilon^{+f(\lambda-t)} [C_3 \cos q(\lambda-t) + C_3' \sin q(\lambda-t)] \\
& + \epsilon^{-f(\lambda+t)} [C_4 \cos q(\lambda+t) + C_4' \sin q(\lambda+t)] \} \quad (89)
\end{aligned}$$

Substituting:

$$\left. \begin{aligned}
C_1^2 + C_2'^2 &= A^2 \\
C_2^2 + C_2'^2 &= B^2 \\
C_3^2 + C_3'^2 &= C^2 \\
C_4^2 + C_4'^2 &= D^2
\end{aligned} \right\} \quad (90)$$

$$\left. \begin{aligned}
\frac{C_1'}{C_1} &= \tan \alpha \\
\frac{C_2'}{C_2} &= \tan \beta \\
\frac{C_3'}{C_3} &= \tan \gamma \\
\frac{C_4'}{C_4} &= \tan \delta
\end{aligned} \right\} \quad (91)$$

gives these equations in the form:

$$\begin{aligned}
i = \epsilon^{-wt} \{ & A \epsilon^{-f(\lambda-t)} \cos [q(\lambda-t) - \alpha] - B \epsilon^{+f(\lambda+t)} \cos [q(\lambda+t) - \beta] \\
& + C \epsilon^{+f(\lambda-t)} \cos [q(\lambda-t) - \gamma] - D \epsilon^{-f(\lambda+t)} \cos [q(\lambda+t) - \delta] \} \quad (92)
\end{aligned}$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-\alpha t} \{ A \epsilon^{-j(\lambda-t)} \cos[q(\lambda-t) - \alpha] + B \epsilon^{+j(\lambda+t)} \cos[q(\lambda+t) - \beta] \\ + C \epsilon^{+j(\lambda-t)} \cos[q(\lambda-t) - \gamma] + D \epsilon^{-j(\lambda+t)} \cos[q(\lambda+t) - \delta] \} \quad (93)$$

In these equations,  $\lambda$  is the distance co-ordinate, using the velocity of propagation as unit distance, and at a transition point from one circuit to another, where the circuit constants change, the velocity of propagation also changes, and so, for the same time, constants  $f$  and  $q$ ,  $h$  and  $k$ , also change, and so  $ku$ , but transformed to the distance variable  $\lambda$ ,  $ku = q\lambda$  remains the same. That is, by introducing the distance variable  $\lambda$ , the distance can be measured throughout the entire circuit, and across transition points, at which the circuit constants change, and the same equations apply throughout the entire circuit. In this case however, in any section of the circuit,

$$\left. \begin{aligned} \lambda &= \sigma_i u \\ &= u \sqrt{L_i C_i} \end{aligned} \right\} \quad (94)$$

where  $L_i$  and  $C_i$  are the inductance and the capacity, respectively, of the section  $i$  of the circuit, per unit length; for instance, per mile.

That is, in a complex circuit, the time variable  $t$  is the same throughout the entire circuit, or in other words, the frequency of oscillation, as represented by  $q$ , and the rate of decay of the oscillation, as represented by the exponential function of time must be the same throughout the entire circuit. Not so however with the distance variable  $u$ : the wave length of the oscillation, and its rate of building up or down, along the circuit, need not be the same, and usually are not, but in some sections of the circuit, the wave length may be far shorter, as in coiled circuits in transformers, due to the higher  $L$ , or in cables, due to the higher  $C$ . To extend the same equations over the entire complex circuit, it therefore becomes necessary to substitute for the distance variable  $u$  another distance variable  $\lambda$  of such character, that the wave length has the same value in all sections of the complex circuit. As the wave length of the section  $i$  is  $\frac{1}{\sqrt{L_i C_i}}$ , this is done by changing the unit distance by the factor:  $\sigma_i = \sqrt{L_i C_i}$ . The distance unit of the new distance

tion  $i$  is  $\frac{1}{\sqrt{L_i C_i}}$ , this is done by changing the unit distance by the factor:  $\sigma_i = \sqrt{L_i C_i}$ . The distance unit of the new distance

variable  $\lambda$  then is the distance traversed by the wave in unit time, hence different in linear measure, for the different sections of the circuit, but offers the advantage of carrying the distance measurements across the entire circuit and over transition points, by the same distance variable  $\lambda$ .

This means, that the length  $u_i$  of any section  $i$  of the complex circuit is expressed by the length:  $\lambda_i = \sigma_i u_i$ .

The introduction of the distance variable  $\lambda$  also has the advantage, that in the determination of the constants  $r$ ,  $L$ ,  $g$ ,  $C$ , of the different sections of the circuit, different linear distance measurements  $u$  may be used. For instance, in the transmission line, the constants may be given per mile, that is, the mile used as unit length, while in a high potential transformer coil as unit of length  $u$  may be used the turn, or the coil, or the total transformer, so that the actual linear length of conductor may be unknown. For instance, choosing the total length of conductor in the high potential transformer as unit length, then the length of the transformer winding in the velocity measure  $\lambda$  is,  $\lambda_0 = \sqrt{L_0 C_0}$  where  $L_0$  = total inductance,  $C_0$  = total electrostatic capacity of transformer.

The introduction of the distant variable  $\lambda$  permits the representation, in the circuit, of apparatus as reactive coils, etc., in which one of the constants is very small compared with the other, and so is usually neglected, and the apparatus considered as "massed inductance" etc., and allows the investigation of the effect of the distributed capacity of reactive coils and similar matters, by representing the reactive coil as a finite (frequently quite long) section of the circuit.

27. Let  $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n$  be a number of transition points, at which the circuit constants change, and the quantities may be denoted by index 1 in the section from  $\lambda_0$  to  $\lambda_1$ , by index 2 in the section from  $\lambda_1$  to  $\lambda_2$ , etc.

At  $\lambda = \lambda_1$ , it then must be:  $i_1 = i_2$ ,  $e_1 = e_2$ ; thus, substituting  $\lambda = \lambda_1$  into equations (92) and (93), and equating, gives two identities, from which it follows:

$$w_2 \pm f_2 = w_1 \pm f_1 \quad (95)$$

that is, since  $w_2 \pm w_1$ , only one of the two component waves can exist. Since these two waves differ from each other only by the sign of  $f$ , by assuming that  $f$  may be either positive or negative, we can select either one of the two waves, and the

equations (92) and (93), or (88) and (89) so assume the simpler forms:

$$i = \epsilon^{-(w+f)t} \{ A \epsilon^{+f\lambda} \cos [q (\lambda - t) - \alpha] - B \epsilon^{-f\lambda} \cos [q (\lambda + t) - \beta] \}$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-(w+f)t} \{ A \epsilon^{+f\lambda} \cos [q (\lambda - t) - \alpha] + B \epsilon^{-f\lambda} \cos [q (\lambda + t) - \beta] \} \quad (96)$$

$$i = \epsilon^{-(w+f)t} \{ \epsilon^{+f\lambda} [A \cos q (\lambda - t) + B \sin q (\lambda - t)] - \epsilon^{-f\lambda} [C \cos q (\lambda + t) + D \sin q (\lambda + t)] \}$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-(w+f)t} \{ \epsilon^{+f\lambda} [A \cos q (\lambda - t) + B \sin q (\lambda - t)] + \epsilon^{-f\lambda} [C \cos q (\lambda + t) + D \sin q (\lambda + t)] \} \quad (97)$$

where  $f$  may be positive or negative.

It is then:

$$w_1 + f_1 = w_2 + f_2 = w_3 + f_3 = \dots = w_n + f_n = m_0 \quad (98)$$

where  $w_1, w_2, w_3$ , etc.,  $w_n$ , are the time constants of the indi-

vidual sections,  $\frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right)$ , and  $m_0$  may be called the *resultant time decrement of the complex circuit*.

The identities:  $i_1 = i_2$  and  $e_1 = e_2$  for  $\lambda = \lambda_1$ , resolved for  $\cos qt$  and  $\sin qt$ , then give 4 equations, by which the integration constants  $A, B, \alpha, \beta$ , or  $A, B, C, D$ , of one section of the circuit are determined by those of the adjoining section, by the expressions:

$$\left. \begin{aligned} A_2 &= \epsilon^{-f_2\lambda_1} \{ a_1 \epsilon^{+f_1\lambda_1} A_1 + b_1 \epsilon^{-f_1\lambda_1} (C_1 \cos 2q\lambda_1 + D_1 \sin 2q\lambda_1) \} \\ B_2 &= \epsilon^{-f_2\lambda_1} \{ a_1 \epsilon^{+f_1\lambda_1} B_1 + b_1 \epsilon^{-f_1\lambda_1} (C_1 \sin 2q\lambda_1 - D_1 \cos 2q\lambda_1) \} \\ C_2 &= \epsilon^{+f_2\lambda_1} \{ a_1 \epsilon^{-f_1\lambda_1} C_1 + b_1 \epsilon^{+f_1\lambda_1} (A_1 \cos 2q\lambda_1 - B_1 \sin 2q\lambda_1) \} \\ D_2 &= \epsilon^{+f_2\lambda_1} \{ a_1 \epsilon^{-f_1\lambda_1} D_1 + b_1 \epsilon^{+f_1\lambda_1} (A_1 \sin 2q\lambda_1 - B_1 \cos 2q\lambda_1) \} \end{aligned} \right\} \quad (99)$$

where:

$$\left. \begin{aligned} a_1 &= \frac{c_1 + c_2}{2 c_2} \\ b_1 &= \frac{c_1 - c_2}{2 c_2} \end{aligned} \right\} \quad (100)$$

$$\left. \begin{aligned} c_1 &= \sqrt{\frac{L_1}{C_1}} \\ c_2 &= \sqrt{\frac{L_2}{C_2}} \end{aligned} \right\} \quad (101)$$

28. The general equation of current and voltage in a complex circuit consists of two terms, a main wave and its reflected wave.

The factor  $\epsilon^{-(w+f)t} = \epsilon^{-m_0 t}$  represents the time decrement, or the decrease of the intensity of the wave with the time, and as such is the same throughout the entire circuit. In an isolated section, of time constant  $w$ , the time decrement, was found however as  $\epsilon^{-w t}$ . That is, with the decrement  $\epsilon^{-w t}$ , the wave dies out in an isolated section at the rate at which its stored energy is dissipated by the power lost in resistance and conductance. In a section of the circuit connected to other sections, the time decrement  $\epsilon^{-m_0 t}$  does not correspond to the power dissipation in the section, that is, the wave does not die out in each section at the rate as given by the power consumed in this section, or in other words, power transfer occurs from section to section, during the oscillation of a complex circuit.

If  $f$  is negative,  $m_0$  is less than  $w$ , and the wave dies out in that particular section at a lesser rate than corresponds to the power consumed in the section, or in other words, in this section of the complex circuit, more power is consumed by  $r$  and  $g$  than is supplied by the decrease of the stored energy, and this section so must receive energy from adjoining sections. Inversely, if  $f$  is positive,  $m_0 > w$ , and the wave dies out more rapidly in that section than its stored energy is consumed by  $r$  and  $g$ ; that is, a part of the stored energy of this section is transferred to the adjoining sections, and only a part—occasionally a very small part—dissipated in the section, and this

section acts as a store of energy for supplying the other sections of the system.

The constant  $f$  of the circuit may be called *energy transfer constant*, and positive  $f$  means transfer of energy from the section to the rest of the circuit, and negative  $f$  means reception of energy from other sections.

This explains the vanishing of  $f$  in a standing wave of a uniform circuit, due to the absence of energy transfer, and its presence in the equations of the traveling wave, in which energy is transferred along the circuit, and in the alternating current equations.

It follows herefrom, that in a complex circuit, some of the  $f$  of the different sections must always be positive, some negative.

In addition to the time decrement,  $\epsilon^{-m_0 t}$ , the waves in equations (96) and (97) also contain the distance decrement  $\epsilon^{+f\lambda}$  for the main wave,  $\epsilon^{-f\lambda}$  for the reflected wave. Negative  $f$  means a decrease of the main wave for increasing  $\lambda$ , or in the direction of propagation, and a decrease of the reflected wave for decreasing  $\lambda$ , that is, also in the direction of propagation, while positive  $f$  means increase of main wave as well as reflected wave, in the direction of propagation along the circuit. That is if  $f$  is negative, and the section so consumes more power than is given by its stored energy, and therefore receives power from the adjoining sections, the electric wave decreases, in the direction of its propagation, showing the gradual dissipation of the power received from adjoining sections. Inversely, if  $f$  is positive and the section so supplies power to adjoining sections, the electric wave increases in this section, in the direction of its propagation.

In other words, in a complex circuit, in sections of low power dissipation, the wave increases, and transfers power to sections of high power dissipation in which the wave decreases.

29. Introducing the resultant time decrement  $m_0$  of the complex circuit, the equations of any section can be expressed by the resultant time decrement of the entire complex circuit,  $m_0$ , and the energy transfer constant of the individual section:

$$f = m_0 - w \quad (102)$$

in the form:

$$i = \epsilon^{-m_0 t} \{ \epsilon^{+f\lambda} [A \cos q(\lambda - t) + B \sin q(\lambda - t)] - \epsilon^{-f\lambda} [C \cos q(\lambda + t) + D \sin q(\lambda + t)] \} \quad (103)$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-m_0 t} \{ \epsilon^{+j\lambda} [A \cos q (\lambda - t) + B \sin q (\lambda - t)] + \epsilon^{-j\lambda} [C \cos q (\lambda + t) + D \sin q (\lambda + t)] \}$$

The constants  $A, B, C, D$ , are the integration constants, and are such as given by the terminal conditions of the problem, as by the distribution of current and voltage in the circuit at the starting moment, for  $t = 0$ , or at one particular point, as  $\lambda = 0$ .

The constants  $m_0$  and  $q$  are dependent upon the circuit conditions: if the circuit is closed upon itself—as usually is the case with an electrical transmission or distribution circuit—and  $\lambda$  the total length of the closed circuit, the equations must for  $\lambda = \lambda$  give the same values as for  $\lambda = 0$ , and therefore  $q\lambda$  must be a complete cycle or a multiple thereof:  $2n\pi$ . That is:

$$q = \frac{2n\pi}{\lambda} \quad (104)$$

and the least value of  $q$ , or the fundamental frequency of oscillation, is:

$$q_0 = \frac{2\pi}{\lambda} \quad (105)$$

and:

$$q = n q_0 \quad (106)$$

If the complex circuit is open at both ends, or grounded at both ends, and so performs a half wave oscillation, and  $\lambda_1 =$  total length of the circuit, it is:

$$\begin{aligned} q_0 &= \frac{\pi}{\lambda_1} \\ q &= n q_0 \end{aligned} \quad (107)$$

and if the circuit is open at one end, grounded at the other end, so performing a quarter wave oscillation, and  $\lambda_2 =$  total length of circuit, it is:

$$\left. \begin{aligned} q_0 &= \frac{\pi}{2\lambda_2} \\ q &= (2n-1) q_0 \end{aligned} \right\} \quad (108)$$



If the length of the complex circuit is very great compared with the frequency of the oscillation,  $q_0$  may have any value. That is, if the wave length of the oscillation is very short compared with the length of the circuit, any wave length, and so any frequency, may occur. With uniform circuits, as transmission lines, this latter case, that is, the response of the line to any frequency, can occur only in the range of very high frequencies. That is, even in a transmission line of several hundred miles length, the lowest frequency of free oscillation is fairly high, and frequencies, which are so high, compared with the fundamental frequency of the circuit, that, considered as higher harmonics thereof, they overlap, must be extremely high, of the magnitude of a million cycles. In complex circuits however, the fundamental frequency may be very much lower, and below machine frequencies, as the velocity of propagation

$\frac{1}{\sqrt{LC}}$  may be quite low in some sections of the circuit, as the

high potential coils of large transformers, and the presence of iron increases the inconstancy of  $L$  for high frequencies, so that in such a complex circuit, even at fairly moderate frequencies, of the magnitude of 10,000 cycles, the circuit may respond to any frequency.

The constant  $m_0$  also is determined by the circuit constants. Upon  $m_0$  depends the energy transfer constant of the circuit section, and thereby the rate of building up, in a section of low power consumption, or building down, in a section of high power consumption. In a closed circuit, however, passing around the entire circuit, the same values of  $e$  and  $i$  must again be reached, and the rates of building up and building down of the wave in the different sections, must therefore be such as to neutralize each other when carried through the entire circuit, that is, the total building up, through the entire complex circuit, must be zero. This gives an equation from which  $m_0$  is determined, as a mean value of the time constants  $w_i$  of the individual circuit sections.

## VII. POWER AND ENERGY OF THE COMPLEX CIRCUIT.

30. The free oscillation of a complex circuit differs from that of the uniform circuit, in that the former contains exponential functions of the distance  $\lambda$ , which represent the shifting, or transfer of power between the sections of the circuit.

Thus the general expression of one term or frequency of

current and voltage in a section of a complex circuit is given by equation:

$$i = \epsilon^{-m_0 t} \{ \epsilon^{+f\lambda} [A \cos q (\lambda - t) + B \sin q (\lambda - t)] - \epsilon^{-f\lambda} [C \cos q (\lambda + t) + D \sin q (\lambda + t)] \} \quad (109)$$

$$e = \sqrt{\frac{L}{C}} \epsilon^{-m_0 t} \{ \epsilon^{+f\lambda} [A \cos q (\lambda - t) + B \sin q (\lambda - t)] + \epsilon^{-f\lambda} [C \cos q (\lambda + t) + D \sin q (\lambda + t)] \}$$

where:

$$q = n q_0$$

$$q_0 = \frac{2\pi}{\lambda}$$

$\lambda$  = total length of circuits, expressed in the distance co-ordinate:

$$\lambda = \sigma u,$$

$u$  being the distance co-ordinate of the circuit section in any measure, as miles, turns, etc., and  $r$ ,  $L$ ,  $g$ ,  $C$ , the circuit constants per unit length of  $u$ , and:

$$\sigma = \sqrt{LC}$$

$$c = \sqrt{\frac{L}{C}}$$

$$w = \frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right) = \text{time constant of circuit section,}$$

$m_0 = w + f$  = resultant time decrement of complex circuit,

$f = m_0 - w$  = energy transfer constant of circuit section.

The instantaneous value of power at any point of the circuit, at any time  $t$ , is:

$$p = e i$$

$$= \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ \epsilon^{+2f\lambda} [A \cos q (\lambda - t) + B \sin q (\lambda - t)]^2 - \epsilon^{-2f\lambda} [C \cos q (\lambda + t) + D \sin q (\lambda + t)]^2 \}$$

$$= \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ [\epsilon^{+2f\lambda} (A^2 + B^2) - \epsilon^{-2f\lambda} (C^2 + D^2)] + [\epsilon^{+2f\lambda} (A^2 - B^2) \cos 2q (\lambda - t) - \epsilon^{-2f\lambda} (C^2 - D^2) \cos 2q (\lambda + t)] + 2 [A B \epsilon^{+2f\lambda} \sin 2q (\lambda - t) - C D \epsilon^{-2f\lambda} \sin 2q (\lambda + t)] \} \quad (110)$$

that is, the instantaneous value of power consists of a constant term, and terms of double frequency in  $(\lambda - t)$  and  $(\lambda + t)$ , that is, in distance  $\lambda$  and time  $t$ .

Integrating (110) over a complete period in time, gives the *effective* or *mean value of power at any point*  $\lambda$ , as:

$$P = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ \epsilon^{+2f\lambda} (A^2 + B^2) - \epsilon^{-2f\lambda} (C^2 + D^2) \} \quad (111)$$

That is, the instantaneous power at any time and any point of the circuit is the difference between the instantaneous power of the main wave, and that of the reflected wave.

The effective power at any point of the circuit is the difference between the effective power of the main wave and that of the reflected wave.

The effective power at any point of the circuit, gradually decreases in any section, with the resultant time decrement of the total circuit,  $\epsilon^{-2m_0 t}$ , and varies gradually or exponentially with the distance  $\lambda$ , the one wave increasing, the other decreasing, so that at one point of the circuit or circuit section the effective power is zero. That is, this point of the circuit is a power node, or point across which no power flows. It is given by:

$$\left. \begin{aligned} &\epsilon^{+2f\lambda} (A^2 + B^2) = \epsilon^{-2f\lambda} (C^2 + D^2) \\ \text{or:} & \end{aligned} \right\} \quad (112)$$

$$\lambda = \frac{1}{4f} \log \frac{C^2 + D^2}{A^2 + B^2}$$

The difference of power between two points of the circuit  $\lambda_1$  and  $\lambda_2$ , that is, the power which is supplied or received (depending upon its sign) by a section  $l = \lambda_2 - \lambda_1$  of the circuit, is given by equation (111) as:

$$P_0 = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ (\epsilon^{+2f\lambda_2} - \epsilon^{+2f\lambda_1}) (A^2 + B^2) - (\epsilon^{-2f\lambda_2} - \epsilon^{-2f\lambda_1}) (C^2 + D^2) \} \quad (113)$$

If  $P_0 > 0$ , this represents the power which is supplied by the section  $l$  to the adjoining sections of the circuit; if  $P_0 < 0$ , this is the power received by the section from the rest of the complex circuit.

If  $f\lambda_2$  and  $f\lambda_1$  are small quantities, the exponential function can be resolved in an infinite series, and all but the first term dropped, as of higher orders, or negligible, and this gives the approximate value:

$$\epsilon \pm {}^{2f}\lambda_2 - \epsilon \pm {}^{2f}\lambda_1 = \pm 2f(\lambda_2 - \lambda_1) = \pm 2fl$$

hence:

$$P_o = fl \sqrt{\frac{L}{C}} \epsilon^{-2m_0 l} \{A^2 + B^2 + C^2 + D^2\} \quad (114)$$

That is, the power transferred from a section of length  $l$ , to the rest of the circuit, or received by the section from the rest of the circuit, is proportional to the length of the section,  $l$ , to its transfer constant  $f$ , and to the sum of the power of main wave and reflected wave.

31. The energy stored by the inductance,  $L$ , of a circuit element  $d\lambda$ , that is, in the magnetic field of the circuit, is:

$$d\eta_1 = \frac{L' i^2}{2} d\lambda$$

where  $L' =$  inductance per unit length of circuit, expressed by the distance co-ordinate  $\lambda$ .

Since  $L =$  inductance per unit length of circuit, of distance co-ordinate  $u$ , and  $\lambda = \sigma u$ , it is:

$$L' = \frac{L}{\sigma} = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}}$$

hence:

$$d\eta_1 = \frac{1}{2} \sqrt{\frac{L}{C}} i^2 d\lambda \quad (115)$$

In general, the circuit constants  $r$ ,  $L$ ,  $g$ ,  $C$ , per unit length,  $u = 1$ , give, per unit length,  $\lambda = 1$ , the circuit constants:

$$\left. \begin{aligned} & \frac{r}{\sigma}; \frac{L}{\sigma}; \frac{g}{\sigma}; \frac{C}{\sigma}; \\ \text{or: } & \frac{r}{\sqrt{LC}}; \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}}; \frac{g}{\sqrt{LC}}; \frac{C}{\sqrt{LC}} = \sqrt{\frac{C}{L}} \end{aligned} \right\} \quad (116)$$

Substituting (109) in equation (115), expanded, and integrated over a complete period in time, gives the effective power stored in the magnetic field at point  $\lambda$ :

$$\frac{d H_1}{d \lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2 m_0 t} \{ [\epsilon^{+2 f \lambda} (A^2 + B^2) + \epsilon^{-2 f \lambda} (C^2 + D^2)] - 2 [(AC - BD) \cos 2 q \lambda + (AD - BC) \sin 2 q \lambda] \} \quad (117)$$

and, integrated over one complete period of distance  $\lambda$ , or one complete wave length, this gives:

$$\frac{d H_1^0}{d \lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2 m_0 t} \{ [\epsilon^{+2 f \lambda} (A^2 + B^2) + \epsilon^{-2 f \lambda} (C^2 + D^2)] \} \quad (118)$$

The energy stored by the inductance  $L$ , or in the magnetic field of the conductor, consists of:

A constant part:

$$\frac{d \eta_0}{d \lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2 m_0 t} \{ \epsilon^{+2 f \lambda} (A^2 + B^2) + \epsilon^{-2 f \lambda} (C^2 + D^2) \} \quad (119)$$

A part which is a function of  $(\lambda - t)$  and  $(\lambda + t)$ :

$$\begin{aligned} \frac{d \eta'}{d \lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2 m_0 t} \{ & [\epsilon^{+2 f \lambda} (A^2 - B^2) \cos 2 q (\lambda - t) + \epsilon^{-2 f \lambda} (C^2 - D^2) \\ & \cos 2 q (\lambda + t)] + 2 [A B \epsilon^{+2 f \lambda} \sin 2 q (\lambda - t) + C D \epsilon^{-2 f \lambda} \sin 2 q (\lambda + t)] \} \end{aligned} \quad (120)$$

A part which is a function of the distance  $\lambda$  only, but not of time  $t$ :

$$-\frac{d \eta''}{d \lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2 m_0 t} \{ (A C - B D) \cos 2 q \lambda + (A D + B C) \sin 2 q \lambda \} \quad (121)$$

And a part which is a function of time  $t$  only, but not of the distance  $\lambda$ :

$$-\frac{d \eta'''}{d \lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2 m_0 t} \{ (A C + B D) \cos 2 q t + (A D - B C) \sin 2 q t \} \quad (122)$$

and the total energy of the electromagnetic field of circuit element  $d\lambda$  at time  $t$  is:

$$\frac{d\eta_1}{d\lambda} = \frac{d\eta_0}{d\lambda} + \frac{d\eta'}{d\lambda} - \frac{d\eta''}{d\lambda} - \frac{d\eta'''}{d\lambda} \quad (123)$$

The energy stored in the electrostatic field of the conductor, or by the capacity  $C$ , is given by:

$$d\eta_2 = \frac{C' e^2}{2} d\lambda$$

or, substituting (116):

$$\frac{d\eta_2}{d\lambda} = \frac{1}{2} \sqrt{\frac{C}{L}} e^2 \quad (124)$$

and, substituting in (124) the value of  $e$ , from equation (109), gives the same expression as (123), except that the sign of the last two terms is reversed. That is:

The total energy of the electrostatic field of circuit element  $d\lambda$  at time  $t$  is:

$$\frac{d\eta_2}{d\lambda} = \frac{d\eta_0}{d\lambda} + \frac{d\eta'}{d\lambda} + \frac{d\eta''}{d\lambda} + \frac{d\eta'''}{d\lambda} \quad (125)$$

and, adding (123) and (125), gives the total stored energy of the electric field of the conductor:

$$\frac{d\eta}{d\lambda} = 2 \left( \frac{d\eta_0}{d\lambda} + \frac{d\eta'}{d\lambda} \right) \quad (126)$$

and, integrated over a complete period of time, this gives:

$$\frac{dH}{d\lambda} = 2 \frac{d\eta_0}{d\lambda} = \frac{1}{2} \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ \epsilon^{+2f\lambda} (A^2 + B^2) + \epsilon^{-2f\lambda} (C^2 + D^2) \} \quad (127)$$

The two last terms,  $\frac{d\eta''}{d\lambda}$  and  $\frac{d\eta'''}{d\lambda}$ , represent the energy which is transferred, or pulsates, between the electromagnetic and the electrostatic field of the circuit; and the term  $\frac{d\eta'}{d\lambda}$  rep-

resents the alternating (or rather oscillating) component of stored energy.

The energy stored by the electric field in a circuit section  $l$ , between  $\lambda_1$  and  $\lambda_2$ , is given by integrating  $\frac{dH}{d\lambda}$  between  $\lambda_2$  and  $\lambda_1$ , as:

$$H = \frac{1}{4} \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ (\epsilon^{+2f\lambda_2} - \epsilon^{+2f\lambda_1}) (A^2 + B^2) - (\epsilon^{-2f\lambda_2} - \epsilon^{-2f\lambda_1}) (C^2 + D^2) \} \quad (128)$$

$$H = \frac{1}{2} l \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ A^2 + B^2 + C^2 + D^2 \}$$

Differentiating (128) after  $t$ , gives the power supplied by the electric field of the circuit, as:

$$P = -\frac{dH}{dt}$$

or, more general:

$$= \frac{m_0}{2} \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ (\epsilon^{+2f\lambda_2} - \epsilon^{+2f\lambda_1}) (A^2 + B^2) - (\epsilon^{-2f\lambda_2} - \epsilon^{-2f\lambda_1}) (C^2 + D^2) \} \quad (129)$$

$$= m_0 l \sqrt{\frac{L}{C}} \epsilon^{-2m_0 t} \{ A^2 + B^2 + C^2 + D^2 \}$$

The power dissipated in the resistance:  $r' d\lambda = \frac{r d\lambda}{\sqrt{LC}}$  and in

the conductance:  $g' d\lambda = \frac{g}{\sqrt{LC}} d\lambda$  of a conductor element  $d\lambda$ ,

and so the total power dissipated in the circuit element  $d\lambda$  is, in the same manner:

$$\frac{d p_1}{d \lambda} = \frac{d p'}{d \lambda} + \frac{d p''}{d \lambda} = 4 w \left( \frac{d \eta_0}{d \lambda} + \frac{d \eta'}{d \lambda} \right) - 4 m \left( \frac{d \eta''}{d \lambda} + \frac{d \eta'''}{d \lambda} \right) \quad (130)$$



where, as before:

$$w = \frac{1}{2} \left( \frac{r}{L} + \frac{g}{C} \right)$$

$$m = \frac{1}{2} \left( \frac{r}{L} - \frac{g}{C} \right)$$

and, integrated over a complete period:

$$\frac{dP_1}{d\lambda} = 4w \frac{d\eta_o}{d\lambda} - 4m \frac{d\eta''}{d\lambda} \quad (131)$$

the power dissipated in the circuit contains a constant term.

$4w \frac{d\eta_o}{d\lambda}$ , and a term which is a periodic function of the distance  $\lambda$ :

$4m \frac{d\eta''}{d\lambda}$ , of double frequency.

Averaged over a half wave of the circuit, or a multiple thereof, the second term eliminates, and it is:

$$\frac{dP_1^0}{d\lambda} = 4w \frac{d\eta_o}{d\lambda}$$

thus, the power dissipated in a section  $l = \lambda_2 - \lambda_1$  of the circuit, is by integrating between limits  $\lambda_1$  and  $\lambda_2$ :

$$P_1^0 = \frac{w_o}{2f} \sqrt{\frac{L}{C}} \epsilon^{-2m_o l} \{ (\epsilon^{+2f\lambda_2} - \epsilon^{+2f\lambda_1}) (A^2 + B^2) + (\epsilon^{-2f\lambda_2} - \epsilon^{-2f\lambda_1}) (C^2 + D^2) \} \quad (132)$$

$$= w \sqrt{\frac{L}{C}} \epsilon^{-2m_o l} \{ A^2 + B^2 + C^2 + D^2 \}$$

Denoting therefore:

$$M^2 = (A^2 + B^2 + C^2 + D^2) \sqrt{\frac{L}{C}} \quad (133)$$

it is:

*Energy stored in the electric field of the circuit section of length  $l$ :*

$$H = \frac{1}{2} l M^2 \epsilon^{-2m_0 t} \quad (134)$$

*Power supplied to the conductor by the decay of the electric field of the circuit:*

$$P = m_0 l M^2 \epsilon^{-2m_0 t} \quad (135)$$

*Power dissipated in the circuit section  $l$ , by its effective resistance and conductance:*

$$P_1^0 = w l M^2 \epsilon^{-2m_0 t} \quad (136)$$

*Power transferred from the circuit section  $l$ , to the rest of circuit:*

$$P_0 = f l M^2 \epsilon^{-2m_0 t} \quad (137)$$

That is:

$w/m_0$  = ratio of power dissipated in the section, to that supplied to the section by the stored energy of its electric field.

$f/m_0$  = fraction of power supplied to the section by its electric field, which is transferred from the section to adjoining sections (or, if  $f < 0$ , received from them).

$f/w$  = ratio of power transferred to other sections, to power dissipated in the section.

$m_0 \div w \div f$  is the ratio of the power supplied to the section by its electric field, dissipated in the section, and transferred from the section to adjoining sections.

These relations obviously are approximate only, and applicable to the case, where the wave length is short.

#### VIII. REFLECTION AND REFRACTION AT TRANSITION POINT.

23. The general equation of the current and voltage in a section of a complex circuit is given by equations (109).

At a transition point  $\lambda_1$  between section 1 and section 2, the constants change by equation (99).

Choosing now the transition point as zero point of  $\lambda$ , so that  $\lambda < 0$  is section 1,  $\lambda > 0$  is section 2, equations (99) assume the form:

$$\left. \begin{aligned} A_2 &= a_1 A_1 + b_1 C_1 \\ B_2 &= a_1 B_1 - b_1 D_1 \\ C_2 &= a_1 C_1 + b_1 A_1 \\ D_2 &= a_1 D_1 - b_1 B_1 \end{aligned} \right\} \quad (138)$$

and herefrom follows:

$$\left. \begin{aligned} c_2 (A_2^2 - C_2^2) &= c_1 (A_1^2 - C_1^2) \\ c_2 (B_2^2 - D_2^2) &= c_1 (B_1^2 - D_1^2) \end{aligned} \right\} \quad (139)$$

If now a wave in section 1,  $A, B$ , travels towards transition point  $\lambda = 0$ , at this point, a part is reflected, giving rise to the reflected wave  $C, D$ , in section 1, while a part is transmitted and appears as main wave,  $A, B$ , in section 2. The wave  $C, D$ , in section 2 would not exist, as it would be a wave coming towards  $\lambda = 0$ , from section 2, and not a part of the wave coming from section 1. In other words, we can consider the circuit as comprising two waves moving in opposite directions:

1. A main wave  $A_1 B_1$ , giving a transmitted wave  $A_2 B_2$  and reflected wave  $C_1 D_1$ .
2. A main wave  $C_2 D_2$ , giving a transmitted wave  $C_1' D_1'$  and reflected wave  $A_2' B_2'$ .

The waves moving towards the transition point are single main waves,  $A_1 B_1$  and  $C_2 D_2$ , and

The waves moving away from the transition point are combinations of waves reflected in the section, and waves transmitted from the other section.

Considering first the main wave moving towards rising  $\lambda$ : in this,  $C_2 = 0 = D_2$ , hence, by (138):

$$\left. \begin{aligned} A_2 &= \frac{2 c_1}{c_1 + c_2} A_1 \\ B_2 &= \frac{2 c_1}{c_1 + c_2} B_1 \end{aligned} \right\} \quad (140)$$

Herefrom it follows:

The *main wave* in section 1:

$$\left. \begin{aligned} i_1 &= \epsilon^{-m_0 t} \epsilon^{+j_1 \lambda} \{A_1 \cos q (\lambda - t) + B_1 \sin q (\lambda - t)\} \\ e_1 &= c_1 \epsilon^{-m_0 t} \epsilon^{+j_1 \lambda} \{A_1 \cos q (\lambda - t) + B_1 \sin q (\lambda - t)\} \end{aligned} \right\} \quad (141)$$

when reaching a transition point  $\lambda = 0$ , resolves into the *reflected wave*, turned back on section 1:

$$\left. \begin{aligned} i_1' &= -\frac{c_2 - c_1}{c_2 + c_1} \epsilon^{-m_0 t} \epsilon^{-f_1 \lambda} \{A_1 \cos q(\lambda + t) - B_1 \sin q(\lambda + t)\} \\ e_1' &= +\frac{c_2 - c_1}{c_2 + c_1} \epsilon^{-m_0 t} \epsilon^{-f_1 \lambda} \{A_1 \cos q(\lambda + t) - B_1 \sin q(\lambda + t)\} \end{aligned} \right\} \quad (142)$$

and the *transmitted wave*, which by passing over the transition point, enters section 2:

$$\left. \begin{aligned} i_2 &= \frac{2 c_1}{c_1 + c_2} \epsilon^{-m_0 t} \epsilon^{+f_2 \lambda} \{A_1 \cos q(\lambda - t) + B_1 \sin q(\lambda - t)\} \\ e_2 &= c_2 \frac{2 c_1}{c_1 + c_2} \epsilon^{-m_0 t} \epsilon^{+f_2 \lambda} \{A_1 \cos q(\lambda - t) + B_1 \sin q(\lambda - t)\} \end{aligned} \right\} \quad (143)$$

the reflection angle,  $\tan(i_1') = -\frac{B_1}{A_1}$ , is supplementary to the impact angle:  $\tan(i_1) = +\frac{B_1}{A_1}$ , and transmission angle:  $\tan(i_2) = +\frac{B_1}{A_1}$ .

Reversing the sign of  $\lambda$  in the equation (142) of the reflected wave, that is, counting the distance for the reflected wave also in the direction of its propagation, and so in opposite direction to the main wave and the transmitted wave, equation (142) becomes:

$$\left. \begin{aligned} i_1'' &= +\frac{c_2 - c_1}{c_2 + c_1} \epsilon^{-m_0 t} \epsilon^{+f_1 \lambda'} \{A_1 \cos q(\lambda' - t) + B_1 \sin q(\lambda' - t)\} \\ e_1'' &= +c_1 \frac{c_2 - c_1}{c_2 + c_1} \epsilon^{-m_0 t} \epsilon^{+f_1 \lambda'} \{A_1 \cos q(\lambda' - t) + B_1 \sin q(\lambda' - t)\} \end{aligned} \right\} \quad (144)$$

and it is then:

$$\begin{array}{lcl}
 i_2 + i_1'' = i_1 & & \\
 \text{or:} & & \\
 i_1 + i_1' = i_2 & & \\
 \frac{c_1}{c_2} e_2 + e_1'' = e_1 & & \\
 e_1 + e_1'' = e_2 & & 
 \end{array}
 \left. \vphantom{\begin{array}{l} i_2 + i_1'' = i_1 \\ i_1 + i_1' = i_2 \\ \frac{c_1}{c_2} e_2 + e_1'' = e_1 \\ e_1 + e_1'' = e_2 \end{array}} \right\} (145)$$

33. That is:

1. In a single electric wave, current and electromotive force are in phase with each other. Phase displacements between current and electromotive force can occur only in resultant waves, that is, in the combination of the main and the reflected wave, and then are a function of the distance  $\lambda$ , as the two waves would travel in opposite directions.

2. When reaching a transition point, a wave splits up into a reflected wave and a transmitted wave, the former returning, in opposite direction, over the same section, the latter entering the adjoining section of the circuit.

3. Reflection and transmission occur without change of the phase angle. That is, the phase of the current, and of the voltage, in the reflected wave and in the transmitted wave, at the transition point is the same as the phase of the main wave or incoming wave. Reflection and transmission with a change of phase angle can occur only by the combination of two waves traveling in opposite direction, over a circuit, that is, in a resultant wave, but not in a single wave.

4. The sum of the transmitted and the reflected current equals the main current, when considering these currents in their respective direction of propagation.

The sum of the voltage of main wave and of reflected wave, equals the voltage of the transmitted wave.

The sum of the voltage of the reflected wave, and of the voltage of the transmitted wave, reduced to the first section by the ratio of voltage transformation  $\frac{c_2}{c_1}$ , equals the voltage of the main wave.

5. At the transition point, so a voltage transformation occurs,

by the factor  $\frac{c_2}{c_1} = \sqrt{\frac{L_2}{C_2} \frac{C_1}{L_1}}$ , that is, the transmitted wave of voltage equals the difference between main wave and reflected wave, multiplied by the transformation ratio  $\frac{c_2}{c_1}$ ;  $e_2 = \frac{c_1}{c_2}(e_1 - e_1'')$ .

As a result thereof, in passing from one section of a circuit to another section, the voltage of the wave may decrease or may increase. If  $\frac{c_2}{c_1} > 1$ , that is, when passing from a section of low inductance and high capacity, into a section of high inductance and low capacity, as from a transmission line into a transformer or a reactive coil, the voltage of the wave is increased; if  $\frac{c_2}{c_1} < 1$ , that is, when passing from a section of high inductance and low capacity, into a section of low inductance and high capacity, as from a transformer to a transmission line, the voltage of the wave is decreased.

This explains the frequent increase to destruction voltages, when entering a station from the transmission line or cable, of an impulse or a wave, which in the transmission line is of relatively harmless voltage.

34. The same equations also apply for a wave passing the transition point in opposite direction, and combining both waves gives the total expressions of current and voltage on the two sides of the transition point.

In the neighborhood of the transition point, that is, for values  $\lambda$ , which are sufficiently small, so that  $\epsilon^{+\lambda}$  and  $\epsilon^{-\lambda}$  can be dropped, this gives the equations:

$$\begin{aligned}
 i_1 &= \epsilon^{-\omega_0 t} \left[ \{A_1 \cos q(\lambda - t) + B_1 \sin q(\lambda - t)\} - \frac{c_2 - c_1}{c_1 + c_2} \{A_1 \cos q(\lambda + t) - B_1 \sin q(\lambda + t)\} - \frac{2c_2}{c_1 + c_2} \{C_2 \cos q(\lambda + t) + D_2 \sin q(\lambda + t)\} \right] \\
 e_1 &= c_1 \epsilon^{-\omega_0 t} \left[ \{A_1 \cos q(\lambda - t) + B_1 \sin q(\lambda - t)\} + \frac{c_2 - c_1}{c_2 + c_1} \{A_1 \cos q(\lambda + t) - B_1 \sin q(\lambda + t)\} + \frac{2c_2}{c_1 + c_2} \{C_2 \cos q(\lambda + t) + D_2 \sin q(\lambda + t)\} \right]
 \end{aligned}
 \tag{146}$$

$$i_2 = -\epsilon^{-m_0 t} \left[ \{C_2 \cos q (\lambda + t) + D_2 \sin q (\lambda + t)\} - \frac{c_1 - c_2}{c_1 + c_2} \{C_2 \cos q (\lambda - t) - D_2 \sin q (\lambda - t)\} - \frac{2c_1}{c_1 + c_2} \{A_1 \cos q (\lambda - t) + B_1 \sin q (\lambda - t)\} \right]$$

$$e_2 = c_2 \epsilon^{-m_0 t} \left[ \{C_2 \cos q (\lambda + t) + D_2 \sin q (\lambda + t)\} + \frac{c_1 - c_2}{c_1 + c_2} \{C_2 \cos q (\lambda - t) - D_2 \sin q (\lambda - t)\} + \frac{2c_1}{c_1 + c_2} \{A_1 \cos q (\lambda - t) + B_1 \sin q (\lambda - t)\} \right]$$

In these equations, the first term is the main wave, the second term its reflected wave, and the third term the wave transmitted from the adjoining section over the transition point.

Expanding and rearranging equations (146) gives as the distance phase angles of the waves:

$$\left. \begin{aligned} \tan i_1 &= \frac{c_2 \{(B_1 - D_2) \cos qt + (A_1 + C_2) \sin qt\}}{(c_1 A_1 - c_2 C_2) \cos qt - (c_1 B_1 + c_2 D_2) \sin qt} \\ \tan i_2 &= \frac{c_1 \{(B_1 - D_2) \cos qt + (A_1 + C_2) \sin qt\}}{(c_1 A_1 - c_2 C_2) \cos qt - (c_1 B_1 + c_2 D_2) \sin qt} \end{aligned} \right\} \quad (147)$$

hence:

$$\frac{\tan i_2}{\tan i_1} = \frac{c_1}{c_2} = \sqrt{\frac{L_1 C_2}{L_2 C_1}} \quad (148)$$

$$\left. \begin{aligned} \tan e_1 &= \frac{(c_1 B_1 + c_2 D_2) \cos qt + (c_1 A_1 - c_2 C_2) \sin qt}{c_2 \{(A_1 + C_2) \cos qt - (B_1 - D_2) \sin qt\}} \\ \tan e_2 &= \frac{(c_1 B_1 + c_2 D_2) \cos qt + (c_1 A_1 - c_2 C_2) \sin qt}{c_1 \{(A_1 + C_2) \cos qt - (B_1 - D_2) \sin qt\}} \end{aligned} \right\} \quad (149)$$

hence:

$$\frac{\tan e_2}{\tan e_1} = \frac{c_2}{c_1} = \sqrt{\frac{L_2 C_1}{L_1 C_2}} \quad (150)$$



That is, at a transition point the distance phase angle of the wave changes, so that the ratio of the tangent functions of the phase angles is constant, and the ratio of the tangent functions of the phase angles of the voltages is proportional, of the currents inversely proportional to the ratio of the circuit constants  $c = \sqrt{\frac{L}{C}}$ .

That is, transition of an electric wave or impulse, from one section of a circuit to another, takes place at a constant ratio of the tangent functions of the phase angle, which ratio is a constant of the circuit sections, between which the transition occurs.

This law is analogous to the law of refraction in optics, except that in the electric wave it is the ratio of the tangent functions, while in optics it is the ratio of the sine functions, which is constant and a characteristic of the media, between which the transition occurs.

This law so may be called the

*Law of refraction of a wave* at the boundary between two circuits, or at a transition point.

The law of refraction of an electric wave at the boundary between two media, that is, at a transition point between two circuit sections, is given by the constancy of the ratio of the tangent functions of the incoming and reflected wave.

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DISCUSSION ON "THE GENERAL EQUATIONS OF THE ELECTRIC CIRCUIT." ATLANTIC CITY, N. J., JULY 2, 1908

(Subject to final revision for the Transactions.)

**Frederick Bedell:** It is unnecessary to dwell upon the great value of the work of Dr. Steinmetz in this and other papers. It might be proper that this paper be presented at one convention and discussed at another, a year or two later. The increasing value of the work of Dr. Steinmetz impresses me more and more, and looking back over his writings of ten or twelve years ago we find that to-day many of them are of even greater value than when they were written.

There is one thing to be noted particularly in this paper, and that is that Dr. Steinmetz has continually called attention to the physical significance and practical meaning of his results. We do not always find this careful rounding out of a paper. Not long ago, in looking over a mathematical paper on a subject in which I was interested, I was much disappointed to find that the paper stopped abruptly, and I was unable to find out what was the author's conclusion.

On account of the nature of this paper of Dr. Steinmetz, we can make only a few casual comments, which bring out nothing new. When we impress an electric impulse upon a circuit, we have a wave going in each direction; each wave keeps going around and around the circuit, until it dies out, and each time it goes around the circuit it gives us one of those terms in the series, which are added up as Mr. Steinmetz has here done.

There is a physical feature to which attention might be called. We are most of us in the habit, when considering an ordinary alternating-current circuit, of thinking that the current in a series circuit is the same in all parts of the circuit, which is correct when there is no distributed capacity. In a circuit with distributed capacity, however, the current is not the same in all parts of the circuit; if we consider some point,  $P$ , in the circuit, the current may be going towards that point from both directions—a condition which we never have in a simple circuit without distributed capacity. This means that we are having electricity accumulated at that point. The point  $P$ , however, is not stationary, but travels along. We can think of this easier by analogy—by an hydraulic analogy. Considering first a pipe which is not flexible, an iron pipe, in which flows water which is not compressible; if we have a certain flow of water in one part of the pipe or circuit we have that same flow in all parts of the circuit. This construction illustrates the circuit without distributed capacity. Now let us consider, instead of a cast-iron pipe, a flexible rubber hose, and cause a series of impulses (as with a plunger) in the hose at one end; as a result of each impulse the hose will swell, and the swelling will travel along. Following the swelling there will be a contraction. Wherever there is a swelling, the water may be flowing from both directions towards the swelling;

and then it immediately flows away from the point and makes the hose contract. There is first a surplus and then a deficit of water (or electricity) and these are transmitted along the hose (or circuit).

Dr. Steinmetz called attention to the rates of decay, one with respect to time and the other with respect to distance. Suppose we have the impulses of different frequencies, or a complex wave which can be analyzed into the different component frequencies. These different frequencies, according to the equations of the paper, will be found to have different rates of decay; this is discussed in Chapters 12 and 13 of Bedell and Crehore's *Alternating Currents*. This means that if we have a complex wave impressed upon the circuit, the different components get to the receiving point at different times; one of them may get there a hundredth part of second later than another, so that when they combine again into the resultant wave the wave form at the receiving end of the circuit and the wave form at the sending end of the circuit will be different. In power transmission this is ordinarily insignificant. When applied to telephony this change in wave form becomes significant, and these equations explain the distortion of speech on telephone lines, a case which has already been worked out in detail by Dr. Pupin.

Dr. Steinmetz speaks of the reflected wave. I would like to ask him whether he considers there is any reflected wave in a uniform circuit, in which there are the positive waves going around the circuit in one direction, and the negative waves going around the circuit in the other direction.

**Dugald C. Jackson:** The equations which Dr. Steinmetz have deduced are finely developed, comprehensively organized offsprings of the equation which was devised by Lord Kelvin in the first days of transatlantic telegraphy to analyze the problem of transmission of electric signal pulses through ocean cables of negligible self-induction. That equation was fertilized and hybridized by Heaviside until it could be applied to the problem of the transmission of telephonic speech-currents in circuits of uniformly distributed resistance, self-induction and capacity. Dr. Steinmetz has taken a third great and important step in developing equations which are applicable to complete circuits composed of sections of diverse characters and which also take adequate account of the terminal conditions or apparatus. The paper deals with the offspring of old friends in the electrical science, but the offspring are much more comprehensive and fully developed than the parents.

It has been usual to refer to a vibrating wire as a mechanical analogue of an electric circuit in either forced or natural electric oscillations,—that is, either when it is required to act under imposed pulsations or when stimulated into action and then left to carry out its own sweet will. A uniform wire however does not afford an illustration of the usual electric circuit of practice.

An electric circuit is ordinarily made up of sections of diverse electrical characters. Some of these sections may be long and some may be short, but they are almost inevitably found in the circuits of practice and their diversity of character robs the uniform wire of its usefulness as an analogue. We must therefore abandon the uniform wire for our analogue and take up a wire composed of sections of diverse character welded end to end. Sections of small resistance may be thought of as vibrating in air, others of higher resistance must be thought of as surrounded by molasses or other liquid of a density to correspond at each section with the desired resistance. In certain sections may be injected a large *mass* or *inertia* to represent high self-induction, and the elasticity from section to section must be adjusted to correspond with the desired electrostatic capacity. When this wire made up of diverse sections of appropriate characters is attached to abutments of suitably adjusted elasticity and inertia it affords a true analogue of electric circuits of practice; but the analogue has become too complex to be illuminating. I have not found it possible to foresee the reactions that will occur in each section when a disturbance of particular character is put on such a wire by external means. Moreover, although uniform vibrating wires or plates, or loaded wires giving the semblance of uniform wires, have been prolifically treated in mathematical works, I have never observed a treatment adequate to predetermine the reactions in any wire of diverse sections. Dr. Steinmetz has now come to us with the formula for the electric circuit itself, which not only makes the analogue of less need but also probably at the same time gives the solution of the analogue.

The paper is reassuring and encouraging to the man who has only looked at it hastily, in that the simpler results set forth in it correspond with the phenomena that we know actually occur. For instance, as far as the telephone circuit is concerned, the effects of "loading" that Dr. Steinmetz sets down categorically as deductions from his equations, are those that are known to occur in the telephone circuit. Thus, it is known that leakage up to a certain degree and under certain conditions assists transmission of speech in a telephone circuit or the transmission of signals through an ocean cable, but that an excess of leakage beyond a point fixed by other conditions makes transmission worse. It is also well known that "loading" a telephone circuit under certain conditions materially improves the transmission of speech, but a line loaded to give admirable transmission with low leakage may afford poorer transmission than an unloaded line if the leakage becomes large. For instance, a particular loaded aerial open-wire telephone line may be good to talk over in clear weather, but let the weather get bad and the leakage become large and the loaded line is frightful to talk over; whereas 100 miles, or 150 miles of loaded underground line gives admirable transmission in all sorts of weather,

because the leakage and other constants of such a line do not vary with the weather. That is the deduction Dr. Steinmetz draws from his equations. He doubtless knew these facts in advance, but they are demonstrated in his equations; and it is assuring to find that the equations do tell the truth in these simple matters, as this truthfulness supports a reliance upon their accuracy in more complex matters.

As an illustration of the fact that this paper carries us beyond what has been heretofore our frontier of knowledge, I will point out that the equations apparently enable us to understand the effects of terminal conditions on the action of circuits. As far as telephone analysis has heretofore gone, it has almost solely related to uniform circuits or to loaded circuits which are brought to the condition of substantially uniform circuits. The equations heretofore have not been able to cope successfully with the difficulties of the complete circuit including the complexities of terminal conditions, and the conditions of the telephone receiver and transmitter have been omitted from mathematical consideration or merely approximated. I believe that great advances in intercommunication by the telephone and telegraph are yet to come about from improvements of terminal conditions, and not so much from further improvements in the knowledge of line conditions. These equations of Dr. Steinmetz apparently lay the foundation for study of terminal conditions that may result in the improvements to which I refer.

I am now going to diverge for a moment and go back to a point to which I referred on Tuesday; and that is the tremendous increase in output of what is often called "pure" science which now accompanies the advance of engineering. This paper illustrates the point. Formerly, engineers were followers of the men of scientific research, and the men of scientific research looked with some disdain upon men of the engineering industries. At the present time the engineers have caught up with and absorbed pretty much all of the accumulated output of the pure scientists, and they quickly absorb the current output as fast as it comes along. To keep the output in pure science (especially in physics and chemistry) abreast with the advance of the engineering industries, the engineers themselves have been forced to enter the field of pure research and their product must be recognized as unexcelled by that of the so-called pure scientists.

Dr. Steinmetz has set out in his paper a number of pronounced facts, and some of them illustrate how truth, when it is presented to one plainly, seems so common-sense and persuasive as to be almost axiomatic. For instance, Dr. Steinmetz calls attention to the fact that "The oscillation of a circuit, which is open at one end, grounded at the other end, is a quarter-wave oscillation, which can contain only the odd harmonics of the fundamental wave of oscillation." He further says "The oscillation of a circuit which is open at both ends, or grounded at both



ends, is a half-wave oscillation, and a half-wave oscillation can also contain the even harmonics of the fundamental wave of oscillation, and so also the constant term, for  $n = 0$  in (78). That is to say, one can get a very erratic wave (compared with a sine alternating wave) when one sets up oscillations, and oscillatory waves need not be always of symmetrical half-pulses. The view that is commonly taught in the classroom explains the facts only partially. In another place there is stated a very interesting and very striking thing. After having studied some familiar equations we find the statement that "the wave shape of the oscillation remains unchanged during the decay of the oscillation." Then another very striking thing is found, where it is pointed out that "the wave does not die out in each section at the rate as given by the power consumed in this section, or in other words, power transfer occurs from section to section, during the oscillation of a complex circuit." I do not think that this had come to my mind before it was pointed out to me by Dr. Steinmetz. It seems a fact of great significance.

There are a number of other similar, clear-cut, categorical statements in the paper, which make it interesting to read even to one who does not undertake to go through the equations one by one for the purpose of getting more in the way of facts out of them.

Finally, I wish to congratulate Dr. Steinmetz upon the directness of the attack with which this paper goes at its object. The studies that have heretofore been made on this subject have been more or less of a roundabout siege. The direct assault, when it can be made, though sometimes imposing greater hardship on the attacking party, ordinarily brings greater results than a siege; or, at least it brings, as a rule, results that are more immediately and broadly useful. Dr. Steinmetz has been able to make successfully a direct attack upon the characteristics of a circuit that varies in character throughout its length. His processes are as novel as they are interesting, and I wish again to congratulate the Institute on receiving this demonstration of the great results that can follow a direct attack by mathematical analysis.

**H. L. Wallau:** I would like to ask Dr. Steinmetz one question. He mentions a voltage transformation at the transition point. If I am not mistaken he told us that at that transition point there could be but one frequency, the same on each side of the point, because the end of each section being the same point as the beginning of the next, the frequency was the same. If a voltage transformation occurs, can there be two voltages at the transition point? How does the transformation take place?

**Chas. P. Steinmetz:** As regards the reflected wave in a uniform circuit, the question Dr. Bedell brought up; if you had a perfectly uniform circuit without terminal points—a closed circuit—we could imagine a wave without reflected wave, going around and around, provided we could produce such a wave. I cannot, how-

ever, at present see any possibility of producing such a wave, since any impulse or any action on the closed circuit would naturally produce two waves running in opposite directions from the starting point, and then either of these could be considered as the main wave, the other as the reflected wave. But in general when dealing with uniform circuits we have terminal points, because although the uniform circuit with terminal points, such as a line open at both ends, is not of very great practical importance, a perfectly uniform circuit without terminal points—that is, the line closed on itself at both ends—does not appear feasible from a practical standpoint.

As to the questions asked by Mr. Thomas, I think there is a misunderstanding. If we consider a circuit, with the generator on one end, the load on the other end, the main wave decreases from the generator toward the load, while the reflected wave decreases from the load toward the generator. That is, looking at the waves in the same direction of the circuit, from the generator toward the load, the one wave decreases, the other increases with the distance. However, in treating of waves increasing or decreasing with the distance, I always consider them in the direction of propagation, which in the reflected wave is opposite to that of the main wave.

Considered in the direction of propagation, the character of the reflected wave thus is always the same as that of the main wave, either increasing with the distance or decreasing with the distance. The two component waves of the general equations, that which decreases with the distance, and that which increases with the distance, thus can not be a main wave and its reflected wave, but each of the two component waves comprises a main wave and its reflected wave.

Regarding the transformation of current and voltage at a transition point, this is a very interesting subject, deserving a far more extended discussion than was feasible in the paper. On the two sides of the transition point the resultant currents must be equal, and so also must be the resultant voltages. These however are the resultant of several waves, the main waves and the reflected waves, and the transformation ratio applies to the individual waves. Thus, at one side of the transition point the main wave and the reflected wave differ from those on the other side of the transition point by a constant factor, the transformation ratio, but at the same time the phase-angle on one side differs from that on the other side in such a manner that the resultant of main and reflected waves is the same at both sides of the transition point. As seen, this feature connects the phase-angles on the two sides of the transition point (hence the law of refraction) with the transformation ratio. For instance, if on one side the main wave and the reflected wave are much greater than on the other side, they are more nearly in opposition on the former side than on the latter, and therefore give the same resultant. However, as the phase-angle

changes from point to point in the circuit, at some point, distant from the transition point, the main wave and the reflected wave are in phase with each other. The resultant is then their sum, and higher on the side where the individual waves are higher. Hence the maximum values, of the wave-crests in the circuits on both sides of the transition point, have to each other the transformation ratio of the transition point (multiplied by the decrement of the circuit between transition point and wave crest). Thus, when transforming up at a transition point, at this point the voltage remains the same, but the higher voltage appears at the next wave-crest.

**W. S. Franklin** (by letter): I have found Mr. Steinmetz's paper on the General Equations of the Electric Circuit to be very instructive, and what I wish to say in discussing the paper relates chiefly to a method for discussing electric-wave motion which is due to Heaviside. Some time ago, in reading a treatise on optics by one of our leading specialists on that subject, I came across this statement: "To prove that any phenomenon is due to wave motion, it is sufficient to show (1) that it is periodic, and (2) that it is propagated with a finite velocity", and immediately there came into my mind the idea of bullets from a Gatling gun, which are periodic and which are propagated with a finite velocity. The fact of the matter is that the idea of periodicity has been too much used in the discussion of wave motion. The wave pulse involves all of the important actions of wave motion, except that action which serves as a basis for the theory of the dispersion of light,\* and the physics of wave motion may be developed in the simplest possible manner by considering the behavior of wave pulses; for it is certainly true, as Heaviside says, that the physics of wave motion is presented in its simplest aspects in a wave pulse, whereas the mathematical formulation of the wave pulse is extremely complicated. On the other hand, the physics of the wave train (a periodic succession of similar waves) is extremely complicated, and its mathematical formulation is fairly simple.

I have recently been engaged in the preparation of a paper on electric-wave motion on transmission lines and on the electric oscillation of transmission lines, and a few abstracts from this paper will serve to illustrate the point of view of Heaviside.

Fig. 1 shows a transmission line, initially open at both ends, one line being positively charged, the other negatively charged—both uniformly. The end of the line is suddenly short-circuited, a current is suddenly established at the end of the line, the charges on the wires disappear, and a *wave sheet*  $W$  travels in the direction of the arrow  $V$ , wiping out the electric field and charges completely and establishing the magnetic field and cur-away from the reader. When the wave sheet  $W$  reaches the

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\* The simple wave trains constituting the parts of the solar spectrum which is formed by a prism are produced by the prism and do not exist *as such* in the complex waves which enter the prism from the sun.



rent as it travels along. The lines of force of the magnetic field are represented by the dots and the magnetic field is directed open end of the line, the energy, which initially was electric, has been wholly converted into magnetic energy; the current, which has the same value over the whole line, is then suddenly reduced to zero at the open end, and, in being reduced to zero, it builds up a reversed charge and a reversed electric field at the extreme end of the line; and a wave sheet travels in a reversed direction along the line, wiping out magnetic field and electric current and laying down the reversed electric field and charge. When this wave sheet reaches the short-circuited end of the line, the line is uniformly charged (with zero current everywhere). The charge then suddenly disappears at the short-circuited end of the line, establishing a reversed current and a reversed magnetic field both of which are established by a wave sheet which travels towards the open end of the line. When this wave sheet reaches the open end of the line, the reversed current is suddenly reduced to zero at the open end, a wave sheet travels back to the short-circuited end and re-establishes the initial charge and initial electric field, so that when this wave sheet has reached the short-circuited end of the line, the entire system is in its initial condition and the line has performed one complete oscillation. Line resistance is ignored in this discussion.

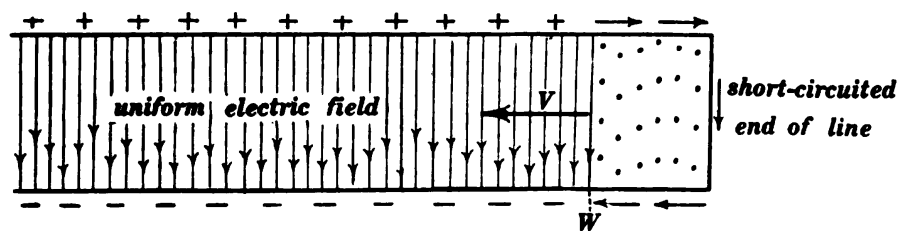


FIG. 1

lished the initial charge and initial electric field, so that when this wave sheet has reached the short-circuited end of the line, the entire system is in its initial condition and the line has performed one complete oscillation. Line resistance is ignored in this discussion.

Fig. 2A represents a generator delivering current over a line of negligible resistance to a receiver of negligible resistance (the system is short-circuited). The receiver end of the line is suddenly opened, the short-circuited current is suddenly reduced to zero at the opened end, and, in being reduced to zero, builds up an electric field between the lines and charges the lines, as shown in Fig. 2B (voltage assumed not to rise to a sufficiently large value to break across from line to line). The short-circuit current (and the magnetic field which corresponds to it) is wiped out by a wave sheet  $W''$ , and the electric field and charges are laid down by this wave sheet which travels towards the generator. A complete cycle of oscillation may easily be traced very much as in the case of Fig. 1.

A battery is suddenly connected to one end of a transmission

line as shown in Fig. 3A, the other end of the transmission line being open. A wave sheet  $W'$  shoots out from the battery end of the line establishing a current in the line and magnetic field between the lines, and also establishing electric charges on the lines and an electric field between them, as shown. When the wave sheet  $W'$  of Fig. 3A reaches the open end of the line, the current in the line is suddenly reduced to zero, and in being reduced to zero builds up a doubled charge and doubled electric field as shown in Fig. 3B, this doubled charge and doubled electric field being established by a wave sheet  $W''$  which travels back towards the battery end of the line. A complete oscillation of the line may easily be traced in this case, as in the case represented in Fig. 1.

A very interesting case is the establishment of current in a long transmission line, short-circuited at the far end, when a

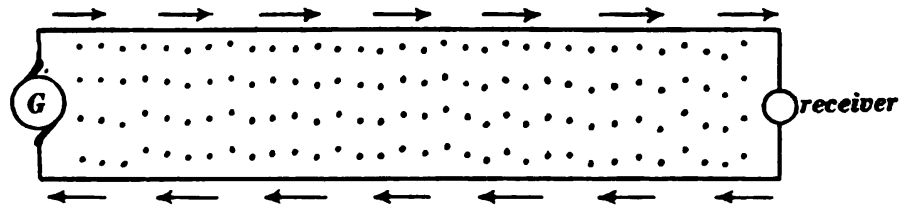


FIG. 2A

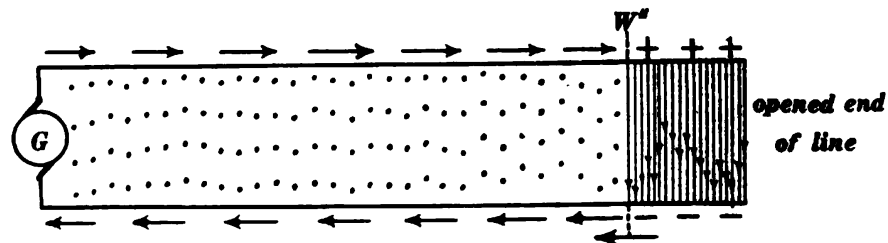


FIG. 2B

battery is connected to the near end, resistance being neglected. A wave sheet shoots out from the battery as represented in Fig.

3A ( $I = E \sqrt{\frac{C}{L}}$ , where  $L$  and  $C$  are line inductance and line capacity per unit of length, respectively,  $E$  is the voltage of the battery, and  $I$  is the current in the line behind the wave sheet).

When this wave sheet reaches the short-circuited end of the line, the electric field between the wires drops to zero at the extreme end and the current rises to  $2I$ , and this doubled current and zero voltage is established over the whole line by a wave sheet which travels back towards the battery end. When this wave sheet reaches the battery end, the whole action is repeated on top of the existing short-circuit current  $2I$ . That is to say, a current  $3I$  and battery voltage  $E$  are established over the whole line by a wave sheet which shoots out from the bat-

tery, and when this wave sheet reaches the short-circuited end of the line, a current  $4I$  and zero voltage are established over the whole line by a wave sheet which travels back towards the battery end. And so on. The action is precisely analogous to the surging motion of a frictionless railroad train which gets under way under a constant locomotive draw-bar pull. At the beginning, the constant pull establishes a certain velocity  $I$  which travels backward until it reaches the caboose; then the stretched coupling springs jerk the caboose forward and a velocity  $2I$  is established which travels forward until the whole train is moving at velocity  $2I$  with coupling springs entirely freed from stress; and then this process is repeated on top of the existing velocity  $2I$ .

A stretched string,  $AB$ , Fig. 4A, is pulled to one side at the

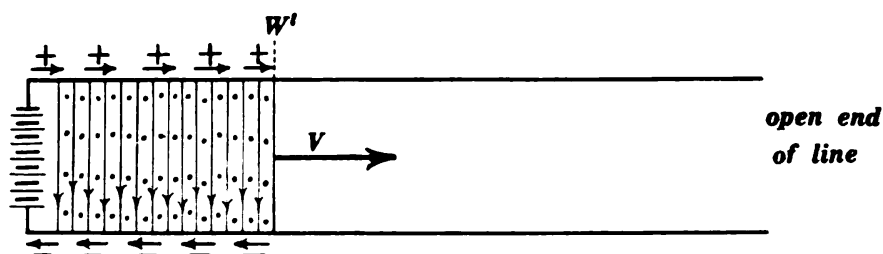


FIG. 3A

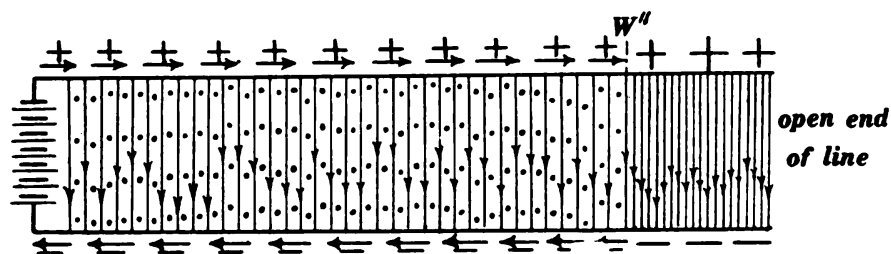


FIG. 3B

point  $P$ ; the water at one end of a canal  $CD$  is held uniformly elevated and the water at the other end uniformly depressed by a gate at the center of the canal; a transmission line,  $EF$ , open at both ends and broken at the middle, has two batteries connected as shown, thus producing an upward electric field between the wires at one end of the line and a downward electric field between the wires at the other end of the line. The three diagrams on Fig. 4B represent the state of affairs when the initial disturbances which are represented in Fig. 4A are released. When the point  $P$  is released the middle portion of the string gets into motion at a uniform velocity and is entirely relieved from the stretch due to the distortion, which is represented in  $AB$ , Fig. 4A. The middle portion of the water in the canal immediately settles to its normal level and is set in uniform motion as represented in the diagram  $C'D'$ . When the switches

in  $EF$ , Fig. 4A, are closed, the charge near the middle of the line disappears, and a uniform current (and magnetic field) is established as shown in the diagram  $E'F'$ . A complete cycle of oscillation may easily be followed in each case.

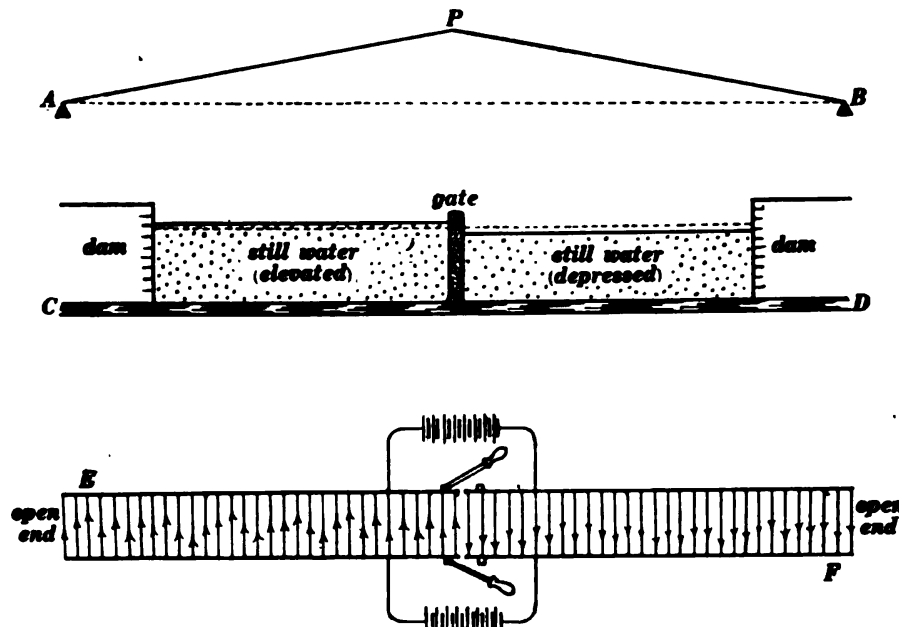


FIG. 4A

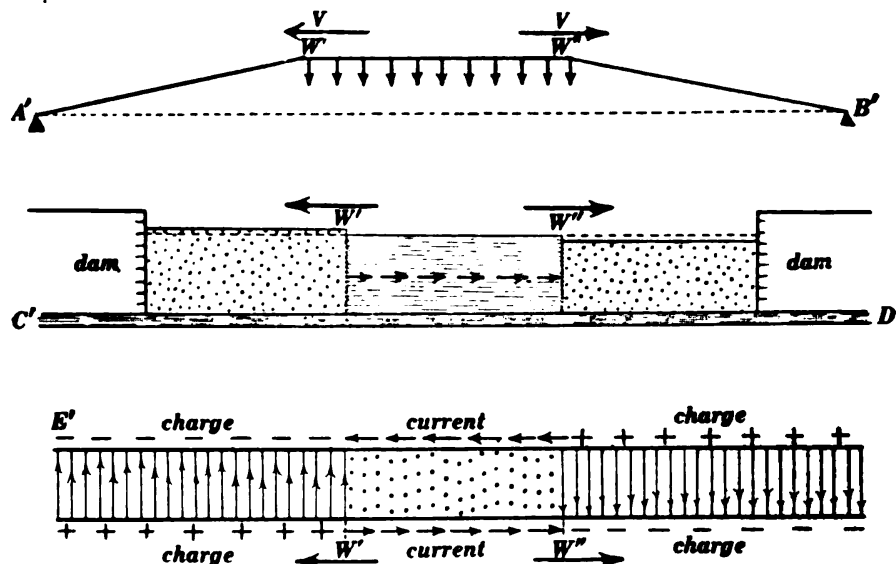


FIG. 4B

The paper which I am preparing on electric-wave motion is devoted (1) to the establishment of clear ideas of the physical action which is involved in the electromagnetic wave; (2) to the discussion of a great variety of special cases; and (3) to the

modification of the results so reached by the effects of line resistance and line leakage. I wish to call attention to the extremely simple discussion of electromagnetic waves which is to be found in the first volume of Heaviside's *Electromagnetic Theory*, pages 306 to 455, in which the fundamental ideas of electric-wave motion are developed on the basis of negligible attenuation (due to resistance of line and leakage between wires), and the effects of line resistance and line leakage are then described in extremely simple terms as a diffusion phenomenon modifying the results that are arrived at in the first instance. I know of no discussion of electric waves more edifying than this of Heaviside's. It would be misleading, however, if I did not mention the fact that throughout the second volume of Heaviside's *Electromagnetic Theory* the mathematical formulation is based, as it must be, upon exponential and periodic functions very much as Dr. Steinmetz has developed it, and that the mathematical formulation of the wave pulse is accomplished by the aid of Fourier's integrals. Such methods must be used if one is to arrive at numerical results, but I think that a clear conception of any physical phenomenon is a thing without which no one can make use of numerical results, or even be led to seek for them.

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